An investigation of the use of continued fractions to solve higher order diophantine equations

Sample Gold Medal Paper
Grade 11

ABSTRACT

We introduce continued fractions and an algorithm to find the continued fractions of rational numbers, and use it to solve linear diophantine equations. Then we introduce an algorithm for finding continued fractions of square roots and use it to investigate solutions to Pell's equation, a quadratic diophantine equation. Finally, we attempt to translate these systems to other higher order diophantine equations, with a cubic diophantine equation. We use a java program (that I wrote) to find the continued fractions after introducing the algorithms.

PROBLEM STATEMENT

A **continued fraction** is a representation of a number *x* in the form shown in figure 1, with integers a_0, $a_{1, \ldots}$ and b_1, $b_{2, \ldots}$.[1] In this paper, we will be specifically working with **simple continued fractions,** which only have numerators of 1, as generalized in figure 2.

$$x = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \ldots}}}, \qquad x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}},$$

Figures 1 and 2. Two generalizations of the continued fraction representation of a number. (http://mathworld.wolfram.com/ContinuedFraction.html)

A continued fraction can also be written in the notation $[a_0; a_1, a_2, a_3, ..]$, which is equivalent to the continued fraction in figure 2.[1] For example, the continued fraction [2; 1, 3, 5] written in fraction notation would be equivalent to 2 + 1/(1 + 1/(3 + 1/5)).

The continued fraction of a rational number can be calculated exactly. If the number is irrational, however, the continued fraction representation will be must be approximated, because the decimal notation of the number is infinite. Continued fractions are used to approximate irrational numbers because of this, but they will never be exact. (Continued fraction expansion was used to approximate pi as 22/7.)

To calculate the continued fraction of a number, first write the fraction as a mixed number. We will use 127/49 in this example. Therefore, this number becomes 2 + 29/49. Next, the fractional part of the number is written as one divided by its reciprocal. We can do this because finding a reciprocal is the same as dividing 1 by the number, therefore dividing 1 by the number twice is equivalent to the original number. Doing this, we are left with 2 + 1/(49/29). By repeating these two steps for every improper fraction we are left with, we can find the continued fraction representation of the number. Doing this, as shown in figure 3, we find that the continued fraction of 127/49 is equal to [2; 1, 1, 2, 4, 1, 1].

$$\frac{127}{49}=2+\cfrac{1}{\frac{49}{29}}=2+\cfrac{1}{1+\frac{20}{29}}=2+\cfrac{1}{1+\cfrac{1}{1+\frac{9}{20}}}=2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\frac{2}{9}}}}=2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{4+\frac{1}{2}}}}}=2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{1+\frac{1}{1}}}}}}$$

Figure 3. The continued fraction expansion of 127/49

The **convergents** of a continued fraction occur when the continued fraction is approximated.[1] In a continued fraction [a_0; a_1, a_2, a_3,…], the *n*th convergent is the value of the fraction from a_0 to a_n. In the continued fraction of 127/49, for example, the 3rd convergent is equal to [2; 1, 1], and the remaining terms of the fraction are neglected. The value of the 3rd convergent would be equal to 2+1/(1+1/(1)), or 5/2. Clearly, this convergent is a rough estimate of the original fraction 127/49.

By increasing the term of convergency, the approximation becomes closer to the original number. The first convergent of the continued fraction of 127/49 is 2, which is very far from the actual value of 2.59184, but the sixth convergent, 70/27 or 2.59259, is clearly much closer to the actual value.

Continued fractions can be used to find solutions of certain types of diophantine equations. **Diophantine equations** are polynomial equations with two or more unknown variables, in which the only solutions are in the set of integers.[2] Linear diophantine equations in the form $ax + by = c$, with integers a, b, and c can be solved using continued fractions.

If these coefficients a and b of a linear diophantine equation were turned into the fraction a/b, the fraction could be rewritten as a continued fraction. For example, with coefficients 47 and

28 (and c = 1), 47/28 becomes the continued fraction [1; 1, 2, 9]. The convergents of this continued fraction written as fractions would then be used to find solutions. Not all of the convergents lead to solutions, however. Each solution must be tested until a solution is found.

In the case of this equation, the third convergent, 5/3, offers a solution to the original diophantine equation, $47x + 28y = 1$. Clearly the fraction itself is not the solution, because we are searching for integer solutions only. The numerator and denominator of the fraction is used and the numbers can be negated if both coefficients are positive or negative. The solution (3, -5) or (-3, 5) could then be used to generate more integer solutions by using the slope of the equations.

Continued fractions can also be applied to solving non-linear diophantine equations. **Pell's equation** is a non-linear diophantine equation of the form $x^2 - ny^2 = c$, with n as an integer that is not a square, and c as an integer[3,4]. A similar continued fraction procedure can be done to find solutions of this equation, which will be explored in another section of this paper.

RELATED RESEARCH

a. Algorithm for Generating Continued Fractions of Rational Numbers

As discussed earlier, there is a simple method for generating the continued fraction representation of a number. This method can be applied to finding approximations of irrational numbers as well, by rounding them to rational numbers. In figure 4, there is a flowchart of the algorithm used to find the continued fraction of a number.

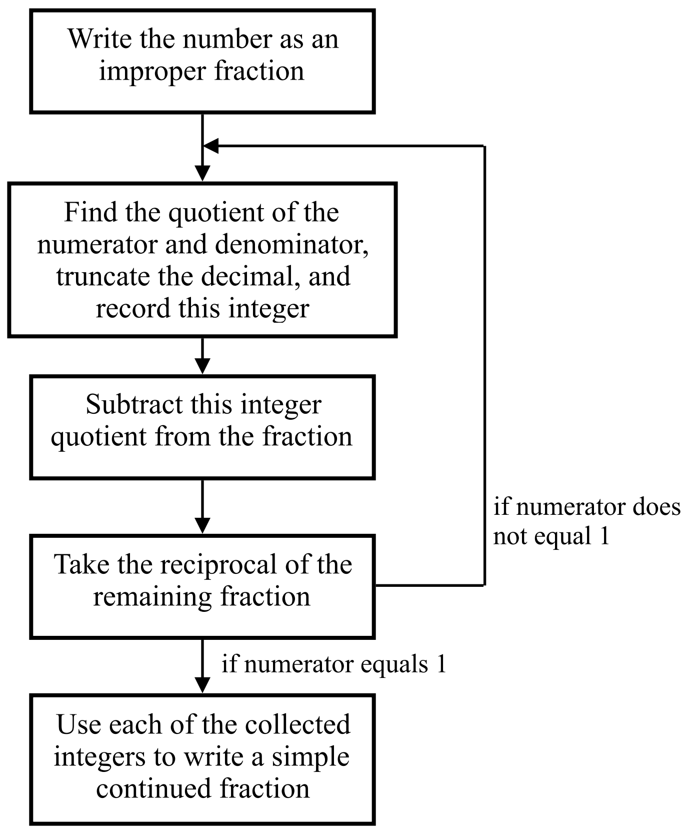

Figure 4. Algorithm for generating continued fractions of rational numbers.

Using this algorithm, we can write a computer program that can generate continued fractions of rational numbers. In java, a do-while loop that repeats iterations until the numerator equals 1 is able to do this to generate the continued fraction of a rational number.

b. Algorithm for Generating Continued Fractions of Square Root Numbers

Generating continued fractions of square roots of numbers is slightly different from the algorithm in figure 4. We will specifically be dealing with irrational numbers with square roots in this algorithm. The continued fractions of square roots of numbers always converge at a repeating section of their continued fraction. The algorithm, shown in figure 5, can be used to find the continued fraction representation of any simple square root, such as $\sqrt{5}$, which becomes $[2; \overline{4}]$

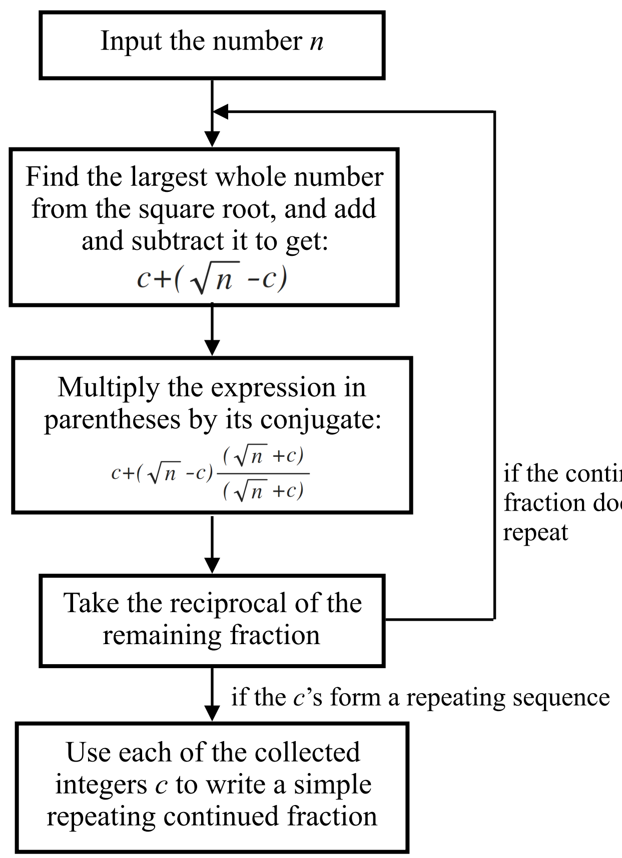

Figure 5. Algorithm for generating continued fractions of square roots of numbers.

Lets use this algorithm to find the continued fraction of √19. First, take out the largest whole number from the square root, and add and subtract it from each side.

$$\sqrt{19} = 4+(\sqrt{19}-4)$$

Next, multiply by the conjugate of the expression in parentheses.

$$\sqrt{19} \;=\; 4+(\sqrt{19}-4)\frac{(\sqrt{19}+4)}{(\sqrt{19}+4)}$$

Simplify.

$$\sqrt{19} \;=\; 4+\frac{3}{(\sqrt{19}+4)}$$

Take the reciprocal of the fraction.

$$\sqrt{19} \;=\; 4+\frac{1}{\frac{\sqrt{19}+4}{3}}$$

Find and take out the largest whole number from the fraction.

$$\sqrt{19} \;=\; 4+\frac{1}{2+\frac{\sqrt{19}+4}{3}-2}$$

Simplify.

$$\sqrt{19} \;=\; 4+\frac{1}{2+\frac{\sqrt{19}-2}{3}}$$

Multiply by the conjugate.

$$\sqrt{19} \;=\; 4+\frac{1}{2+\frac{\sqrt{19}-2}{3}\,\frac{(\sqrt{19}+2)}{(\sqrt{19}+2)}}$$

Simplify and take the reciprocal of the fraction.

$$\sqrt{19} \;=\; 4+\frac{1}{2+\frac{1}{\frac{\sqrt{19}+2}{5}}}$$

As this algorithm progresses, it eventually converges at $\sqrt{19} \;=\; [4;\overline{2,1,3,1,2,8}]$. This algorithm can also be translated into a computer program, although a more complex one than the simpler algorithm for rational continued fractions.

c. Convergents of a Continued Fraction

In this section, we will derive a formula for the convergents of a continued fraction. To do this, we will use induction. Let's start with the simple continued fraction [a_0; a_1, a_2, a_3, a_4,…. a_x]. A convergent of a continued fraction occurs when the value of the fraction is calculated from a_0 to a_n, when *n* is between 0 and *x*. Therefore, we can define the *n*th convergent as $\frac{p_n}{q_n}$, in which p_n and q_n are the numerator and denominator of the convergent, respectively.

For *n* = 1, we have [a_0; a_1].

$$\frac{p_1}{q_1} = a_0 + \frac{1}{a_1}$$

Combine.

$$\frac{p_1}{q_1} = \frac{a_1 a_0 + 1}{a_1}$$

Now, let's repeat this for *n* = 2.

$$\frac{p_2}{q_2} = a_0 + \frac{1}{a_1 + \frac{1}{a_2}} = a_0 + \frac{1}{\frac{a_2 a_1 + 1}{a_2}} = \frac{a_2 a_1 a_0 + a_2 + a_0}{a_2 a_1 + 1}$$

If we repeat this process for the first six convergents, we can make a chart of *n*, p_n, and q_n.

Table 1. p_n and q_n for the first six convergents of a continued fraction.

n	p_n	q_n
0	a_o	1
1	$a_1a_0 + 1$	a_1
2	$a_2a_1a_0 + a_2 + a_0$	$a_2a_1 + 1$
3	$a_3a_2a_1a_0 + a_3a_2 + a_3a_0 + a_1a_0 + 1$	$a_3a_2a_1 + a_3 + a_1$
4	$a_4a_3a_2a_1a_0 + a_4a_3a_2 + a_4a_3a_0 + a_4a_1a_0 + a_4 + a_2 + a_0$	$a_4a_3a_2a_1 + a_4a_3 + a_4a_1 + a_2a_1 + 1$
5	$a_5a_4a_3a_2a_1a_0 + a_5a_4a_3a_2 + a_5a_4a_3a_0 + a_5a_4a_1a_0 + a_5a_2a_1a_0 + a_3a_2a_1a_0 + a_5a_4 + a_3a_0 + a_1a_0 + 1$	$a_5a_4a_3a_2a_1 + a_5a_4a_3 + a_5a_4a_1 + a_5a_2a_1 + a_5 + a_3 + a_1$

With these values in table 1, we can already notice a pattern. Every p_n and p_{n-2}, and q_n and q_{n-2} share similar terms. First, let's subtract these terms for the numerators, as shown in figure 6, to try to find a more concrete pattern.

$$p_2 - p_0 = a_2a_1a_0 + a_2 = a_2(a_1a_0 + 1) \qquad = a_2p_1$$

$$p_3 - p_1 = a_3a_2a_1a_0 + a_3a_2 + a_3a_0$$

$$= a_3(a_2a_1a_0 + a_2 + a_0) \qquad = a_3p_2$$

$$p_4 - p_2 = a_4a_3a_2a_1a_0 + a_4a_3a_2 + a_4a_3a_0 + a_4a_1a_0 + a_4$$

$$= a_4(a_3a_2a_1a_0 + a_3a_2 + a_3a_0 + a_1a_0 + 1). \qquad = a_4p_3$$

Figure 6. Subtracting p_{n-2} from p_n.[6]

Based on figure 6, it appears that p_n - p_{n-2} = a_n p_{n-1}. By subtracting p_n from both sides, we find that p_n = a_n p_{n-1} + p_{n-2}. To find q_n, we can perform a similar procedure to find that the formula seems to be q_n = a_n q_{n-1} + q_{n-2}. However, we have not proved this formula yet.

The first step of a proof by induction is to prove that the formula works for n = 1. We already know that the first convergent of the continued fraction [a_0; a_1, a_2, a_3, a_4,…, a_x] is equal to $\frac{p_1}{q_1} = \frac{a_1a_0+1}{a_1}$, and since $p_0 = a_0$, $p_{-1} = 1$, $q_0 = 1$, and $q_{-1} = 0$, this is equal to the formula that we are trying to prove by substitution.

Next we assume our formula is true for n = k.

$$\frac{p_k}{q_k} = \frac{a_kp_{k-1}+p_{k-2}}{a_kq_{k-1}+q_{k-2}}$$

Finally, we show that the formula holds true for n = k + 1. Since

$$[a_0; a_1, a_2, a_3, a_4, \ldots, a_k, a_{k+1}] = [a_0; a_1, a_2, a_3, a_4, \ldots, a_k + a_{k+1}^{-1}],$$

we can show that the formula is true by using this to substitute $a_k + a_{k+1}^{-1}$ for a_{k+1}.

$$\frac{p_{k+1}}{q_{k+1}} = \frac{\left(a_k + \frac{1}{a_{k+1}}\right)p_{k-1} + p_{k-2}}{\left(a_k + \frac{1}{a_{k+1}}\right)q_{k-1} + q_{k-2}}$$

Multiply by a_{n+1} and distribute.

$$\frac{p_{k+1}}{q_{k+1}} = \frac{a_{k+1}a_k p_{k-1} + p_{k-1} + a_{k+1}p_{k-2}}{a_{k+1}a_k q_{k-1} + q_{k-1} + a_{k+1}q_{k-2}}$$

Factor out a a_{n+1}.

$$\frac{p_{k+1}}{q_{k+1}} = \frac{a_{k+1}(a_k p_{k-1} + p_{k-2}) + p_{k-1}}{a_{k+1}(a_k q_{k-1} + q_{k-2}) + q_{k-1}}$$

Substitute p_k and q_k because $p_n = a_n p_{n-1} + p_{n-2}$ and $q_n = a_n q_{n-1} + q_{n-2}$.

$$\frac{p_{k+1}}{q_{k+1}} = \frac{a_{k+1}p_k + p_{k-1}}{a_{k+1}q_k + q_{k-1}}$$

Therefore, this formula finds the *n*th convergent of any continued fraction, because we proved that the formula works for $n = 1$ and every integer after $n = 1$.

d. Solving Pell's Equation

Pell's equation is a non-linear diophantine equation of the form $x^2 - ny^2 = c$, with n as a non-square integer and c as an integer[3,4]. For example, the equation $x^2 - 4y^2 = 1$ is not a form of Pell's equation because 4 is a square. This restriction is applied because integer solutions would be simple to find by simply factoring the equation. In this section, we will be exploring forms of Pell's equation with $c = 1$.

Solving Pell's equation with continued fractions is very similar to solving linear diophantine equations, which we briefly explored earlier. The first step is to find the continued fraction of the square root of *n*, the coefficient of the y^2 term. Next, we will test the convergents of this repeating continued fraction to find solutions of the equation.

Lets start by exploring the equation $x^2 - 7y^2 = 1$. To find solutions, we will first use the algorithm to find the continued fraction of √7. After applying the algorithm, we find that √17 = [2; 1, 1, 1, 4].

Next, we need to find the convergents of the continued fraction and test for solutions. However, rather than evaluating each convergent by hand each time, which can become time consuming, we can use the formula that we just derived, shown in figure 7, to find the convergents, in which p_n and q_n are the numerator and denominator of the *n*th term, respectively, and a_n is the *n*th term of the continued fraction[5].

$$\frac{p_n}{q_n} = \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}}$$

Figure 7. Formula for the convergents of a continued fraction.

We can use this equation to find the first few convergents of this continued fraction, to test solutions. Table 2 shows the first 5 convergents of the continued fraction of √7, and the results of tests for solutions.

As shown in Table 2, the solutions of the equation $x^2 - 7y^2 = 1$ were in the fourth convergent. We were able to negate the x and y coordinates because since each of the numbers is being squared, the negatives have no effect on the results of the equation.

Table 2. Testing for solutions of x^2 - $7y^2 = 1$

n	Convergent	Solution	(x, y)
1	$\frac{2}{1}$	no	
2	$\frac{3}{1}$	no	
3	$\frac{5}{2}$	no	
4	$\frac{8}{3}$	yes	$(8,3), (-8, -3), (-8,3), (8, -3)$
5	$\frac{37}{14}$	no	

We were able to solve linear diophantine equations and Pell's equation using continued fractions, so perhaps we can solve higher order diophantine equations using a similar algorithm. Lets try this with a cubic equation, in a similar form to Pell's equation, $x^3 - ny^3 = 1$, where n is any non-cube integer. Lets try this with $x^3 - 7y^3 = 1$.

First, we must find the continued fraction of the coefficient. However, instead of using the square root like we did when solving Pell's equation, we must use the cube root of n, because it is a cubic equation. However, we do not have an algorithm for cube root irrationals; only square roots, so we must first investigate this.

e. Investigating Cube Root Continued Fractions

Let's attempt to find the continued fraction of, for example, cube root of 2 by using our algorithm for square roots of integers.

First, lets take out the greatest whole number from the cube root.

$$\sqrt[3]{2} = 1 + (\sqrt[3]{2} - 1)$$

Multiply by the conjugate of the expression in parentheses.

$$\sqrt[3]{2} = 1 + (\sqrt[3]{2} - 1)\frac{(\sqrt[3]{2} + 1)}{(\sqrt[3]{2} + 1)}$$

Distribute and simplify.

$$\sqrt[3]{2} = 1 + \frac{(\sqrt[3]{4} - 1)}{(\sqrt[3]{2} + 1)}$$

This method does not appear to be able to generate a continued fraction for a cube root expression, because we are left with radicals in both the numerator and denominator. This is because multiplying the conjugate of a cube root expression does remove the radical in this situation. There is nothing that we can multiply in this situation to remove the cube root from the numerator of the expression.

We can try to approximate the cube root of *n* using a decimal approximation of the value, being that we are unable to determine the exact value. To do this, we can use an accurate decimal approximation of, as in our example, the cube root of 2. Because of the java program that we wrote for the algorithm for finding continued fractions of rational numbers, we can use a long, and therefore more accurate, decimal approximation of the number. Since our algorithm is for fractions, we simply need to turn the decimal into a fraction, by dividing the decimal by its order of magnitude. After inputing this into the program, we find that:

$$\sqrt[3]{2} = 1.259921049 = \frac{1259921049}{1000000000} = [1;3,1,5,1,1,4,1,1,8,1,22,1,1,1,1,1,1,7,44,2]$$

However, this continued fraction does not appear to repeat or even begin to converge at a repeating series. Furthermore, even though we used a very accurate decimal approximation, we should not look at the last half of the continued fraction, because the small changes led to a larger inaccuracy. To see this clearer, we can compare the exact continued fraction of a square root and an approximation of the same number. The exact continued fraction of √19, for example, is $\sqrt{19} = [4;\overline{2,1,3,1,2,8}]$, but using the decimal approximation, it is 4.3588989 = [4; 2, 1, 3, 1, 2, 8, 2, 1, 4, 1, 2, 1, 9, 1, 3, 1, 3, 2], which is only accurate for the first 9 digits before becoming garbled. However, there is a solution to Pell's equation with $n = 19$ in these first few numbers, so it is possible that there is a solution in the early convergents of the cubic approximation.

f. Attempting to use Approximated Continued Fractions to Solve Cubic Diophantine Equations

Despite the inaccuracy of the end of our continued fraction of the cube root of 2, we can try to use its convergents to find solutions to our cubic diophantine equation, $x^3 - 7y^3 = 1,$ to find integer solutions. The approximate continued fraction of an the cube root of 7 is:

$$\sqrt[3]{7} = 1.91293118 = \frac{191293118}{100000000} = [1;1,10,2,16,2,1,4,2,3,3,1,1,1,3,1,3,1,1,2]$$

We find the convergents using our formula, and then test each in the equation, shown in table 3.

Table 3. Testing for solutions of $x^3 - 7y^3 = 1$

n	Convergent	Test	Solution	(x, y)
1	$\frac{1}{1}$	$1^3 - 7(1)^3 = -6$	no	
2	$\frac{2}{1}$	$2^3 - 7(1)^3 = 1$	yes	$(2,1)$
3	$\frac{21}{11}$	$21^3 - 7(11)^3 = -56$	no	
4	$\frac{44}{23}$	$44^3 - 7(23)^3 = 15$	no	
5	$\frac{725}{379}$	$725^3 - 7(379)^3 = -1448$	no	
6	$\frac{1494}{781}$	$1494^3 - 7(781)^3 = 4997$	no	
7	$\frac{2219}{1160}$	$2219^3 - 7(1160)^3 = -2541$	no	

Based on table 3, the continued fraction for the cube root of 7 generated a solution to our cubic diophantine equation, $x^3 - 7y^3 = 1$, at (2, 1). Once we reached higher convergents, the inaccuracy due to the approximation of our continued fraction shows, as the tests in the equation become increasingly farther from the desired value of 1.

CONCLUSION

We used continued fractions to

- solve linear diophantine equations,
- Pell's equation, and
- investigated other higher order diophantine equations

Higher order diophantine equations should be investigated further with

- the use of continued fractions to solve equations with degrees higher than three and
- an algorithm for finding continued fractions of irrational numbers such as cube roots should be investigated

Works Cited

1. "Continued Fraction." -- from Wolfram MathWorld. N.p., n.d. Web. 22 Nov. 2015.
2. "Diophantine Equation." -- from Wolfram MathWorld. N.p., n.d. Web. 22 Nov. 2015.
3. Mollin, R. A. "Simple Continued Fraction Solutions For Diophantine Equations*." *Expositiones Mathematicae*. 1st ed. Vol. 19. N.p.: n.p., 2001. 55-73. *ScienceDirect*. Web. 20 Oct. 2015.
4. "Pell Equation." -- from Wolfram MathWorld. N.p., n.d. Web. 22 Nov. 2015.
5. Yang, Seung Hyun. "Continued Fractions and Pell's Equation." (n.d.): n. pag. Web. 6 Dec. 2015.
6. "The Topsy-Turvy World of Continued Fractions [online]." (n.d.): 410-41. Web. 7 Dec. 2015.
7. Hooper, Dan. "Continued Fractions." (2009): n. pag. Web. 7 Dec. 2015.
8. "Continued Fractions and Characteristic Recurrences." <i>MathPages</i>. N.p., n.d. Web. 12 Dec. 2015.