MATHEMATICAL ASSOCIATION OF AMERICA

Solutions Pamphlet

American Mathematics Competitions

14th Annual

AMC 10 A

American Mathematics Contest 10 A

Tuesday, February 5, 2013

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, email, internet or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

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The problems and solutions for this AMC 10 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 10 Subcommittee Chair:

Dr. Leroy Wenstrom

1. **Answer (C):** A 5-mile taxi ride costs $\$1.50 + 5(\$0.25) = \$2.75$.

2. **Answer (B):** Filling the cup 4 times will give Alice 1 cup of sugar. To get $2\frac{1}{2}$ cups of sugar, she must fill it $4 + 4 + \frac{1}{2} \cdot 4 = 10$ times.

3. **Answer (E):** The legs of $\triangle ABE$ have lengths $AB = 10$ and BE. Therefore $\frac{1}{2} \cdot 10 \cdot BE = 40$, so $BE = 8$.

4. **Answer (C):** The softball team could only have scored twice as many runs as their opponent when they scored an even number of runs. In those games their opponents scored

$$\frac{2}{2} + \frac{4}{2} + \frac{6}{2} + \frac{8}{2} + \frac{10}{2} = 15 \text{ runs.}$$

In the games the softball team lost, their opponents scored

$$(1 + 1) + (3 + 1) + (5 + 1) + (7 + 1) + (9 + 1) = 30 \text{ runs.}$$

The total number of runs scored by their opponents was $15 + 30 = 45$ runs.

5. **Answer (B):** The total shared expenses were $105+125+175 = 405$ dollars, so each traveler's fair share was $\frac{1}{3} \cdot 405 = 135$ dollars. Therefore $t = 135-105 = 30$ and $d = 135 - 125 = 10$, so $t - d = 30 - 10 = 20$.

OR

Because Dorothy paid 20 dollars more than Tom, Sammy must receive 20 more dollars from Tom than from Dorothy.

6. **Answer (D):** The 5-year-old and the two brothers who went to play baseball account for three of the four brothers who are younger than 10. Because the only age pairs that sum to 16 are 3 and 13, 5 and 11, and 7 and 9, the brothers who went to the movies must be 3 and 13 years old. Hence the 7-year-old and 9-year-old brothers went to play baseball, and Joey is 11.

7. **Answer (C):** Because English is required, the student must choose 3 of the remaining 5 courses. If the student takes both math courses, there are 3 possible choices for the final course. If the student chooses only one of the 2 possible

math courses, then the student must omit one of the 3 remaining courses, for a total of $2 \cdot 3 = 6$ programs. Hence there are $3 + 6 = 9$ programs.

OR

Because English is required, there are 5 remaining courses from which a student must choose 3. Of those $\binom{5}{3}$ possibilities, one does not include a math course. Thus the number of possible programs is $\binom{5}{3} - 1 = 9$.

8. **Answer (C):** Factoring 2^{2012} from each of the terms and simplifying gives

$$\frac{2^{2012}(2^2+1)}{2^{2012}(2^2-1)} = \frac{4+1}{4-1} = \frac{5}{3}\,.$$

9. **Answer (B):** If Shenille attempted x three-point shots and $30 - x$ two-point shots, then she scored a total of $\frac{20}{100} \cdot 3 \cdot x + \frac{30}{100} \cdot 2 \cdot (30 - x) = 18$ points.

   Remark: The given information does not allow the value of x to be determined.

10. **Answer (E):** Because six tenths of the flowers are pink and two thirds of the pink flowers are carnations, $\frac{6}{10} \cdot \frac{2}{3} = \frac{2}{5}$ of the flowers are pink carnations. Because four tenths of the flowers are red and three fourths of the red flowers are carnations, $\frac{4}{10} \cdot \frac{3}{4} = \frac{3}{10}$ of the flowers are red carnations. Therefore $\frac{2}{5} + \frac{3}{10} = \frac{7}{10} = 70\%$ of the flowers are carnations.

11. **Answer (A):** Let n be the number of student council members. Because there are 10 ways of choosing the two-person welcoming committee, it follows that $10 = \binom{n}{2} = \frac{1}{2}n(n-1)$, from which $n = 5$. The number of ways to select the three-person planning committee is $\binom{5}{3} = 10$.

12. **Answer (C):** Because $\overline{EF}$ is parallel to $\overline{AB}$, it follows that $\triangle FEC$ is similar to $\triangle ABC$ and $FE = FC$. Thus half of the perimeter of $ADEF$ is $AF + FE = AF + FC = AC = 28$. The entire perimeter is 56.

13. **Answer (B):** Each such three-digit number must have the form aba, where a and b are digits and $a \neq 0$. Such a number will not be divisible by 5 if and only if $a \neq 5$. If a is equal to 1, 2, 3, or 4, then any of the ten choices for b satisfies the requirement. If a is equal to 6, 7, 8, or 9, then there are 8, 6, 4, or 2 choices for b, respectively. This results in $4 \cdot 10 + 8 + 6 + 4 + 2 = 60$ numbers.

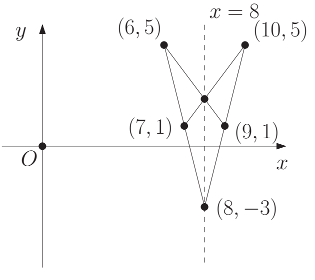
14. **Answer (D):** The large cube has 12 edges, and a portion of each edge remains after the 8 small cubes are removed. All of the 12 edges of each small cube are also edges of the new solid, except for the 3 edges that meet at a vertex of the large cube. Thus the new solid has a total of $12 + 8(12 - 3) = 84$ edges.

15. **Answer (D):** Denote the length of the third side as x, and the altitudes to the sides of lengths 10 and 15 as m and n, respectively. Then twice the area of the triangle is $10m = 15n = \frac{1}{2}x(m + n)$. This implies that $m = \frac{3}{2}n$, so

$$15n = \frac{1}{2}x\left(\frac{3}{2}n + n\right) = \frac{5}{4}xn\,.$$

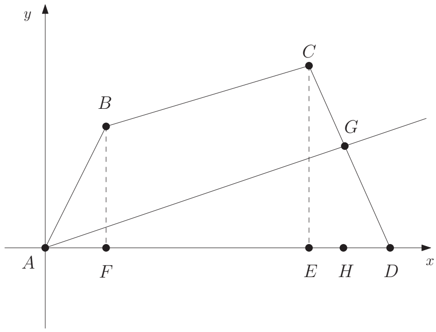
Therefore $15 = \frac{5}{4}x$, and $x = 12$.

16. **Answer (E):** The reflected triangle has vertices $(7, 1)$, $(8, -3)$, and $(10, 5)$. The point $(9, 1)$ is on the line segment from $(10, 5)$ to $(8, -3)$. The line segment from $(6, 5)$ to $(9, 1)$ contains the point $(8, \frac{7}{3})$, which must be on both triangles, and by symmetry the point $(7, 1)$ is on the line segment from $(6, 5)$ to $(8, -3)$. Therefore the union of the two triangles is also the union of two congruent triangles with disjoint interiors, each having the line segment from $(8, -3)$ to $(8, \frac{7}{3})$ as a base. The altitude of one of the two triangles is the distance from the line $x = 8$ to the point $(10, 5)$, which is 2. Hence the union of the triangles has area $2 \cdot (\frac{1}{2} \cdot 2 \cdot (\frac{7}{3} + 3)) = \frac{32}{3}$.



17. **Answer (B):** Alice and Beatrix will visit Daphne together every $3 \cdot 4 = 12$ days, so this will happen $\lfloor \frac{365}{12} \rfloor = 30$ times. Likewise Alice and Claire will visit together $\lfloor \frac{365}{3 \cdot 5} \rfloor = 24$ times, and Beatrix and Claire will visit together $\lfloor \frac{365}{4 \cdot 5} \rfloor = 18$ times. However, each of these counts includes the $\lfloor \frac{365}{3 \cdot 4 \cdot 5} \rfloor = 6$ times when all three friends visit. The number of days that exactly two friends visit is $(30 - 6) + (24 - 6) + (18 - 6) = 54$.

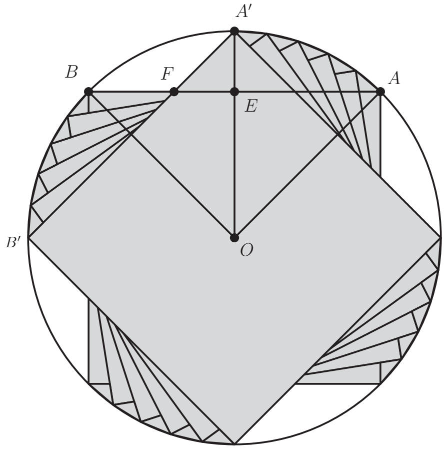
18. **Answer (B):** Let line AG be the required line, with G on $\overline{CD}$. Divide $ABCD$ into triangle ABF, trapezoid $BCEF$, and triangle CDE, as shown. Their areas are 1, 5, and $\frac{3}{2}$, respectively. Hence the area of $ABCD = \frac{15}{2}$, and the area of triangle $ADG = \frac{15}{4}$. Because $AD = 4$, it follows that $GH = \frac{15}{8} = \frac{r}{s}$. The equation of $\overline{CD}$ is $y = -3(x-4)$, so when $y = \frac{15}{8}$, $x = \frac{p}{q} = \frac{27}{8}$. Therefore $p+q+r+s = 58$.



19. **Answer (C):** For the base-b representation of 2013 to end in the digit 3, the base b must exceed 3. Also, b must divide $2013-3 = 2010$, so b must be one of the 16 positive integer factors of $2010 = 2 \cdot 3 \cdot 5 \cdot 67$. Thus there are $16-3 = 13$ bases in which 2013 ends with a 3.

20. **Answer (C):** Let O be the center of unit square $ABCD$, let A and B be rotated to points A' and B', and let $\overline{OA'}$ and $\overline{A'B'}$ intersect $\overline{AB}$ at E and F, respectively. Then one quarter of the region swept out by the interior of the square consists of the $45°$ sector AOA' with radius $\frac{\sqrt{2}}{2}$, isosceles right triangle OEB with leg length $\frac{1}{2}$, and isosceles right triangle $A'EF$ with leg length $\frac{\sqrt{2}-1}{2}$. Thus the area of the region is

$$4\left(\left(\frac{\sqrt{2}}{2}\right)^2\left(\frac{\pi}{8}\right)+\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}-1}{2}\right)^2\right) = 2-\sqrt{2}+\frac{\pi}{4}.$$



21. **Answer (D):** For $1 \le k \le 11$, the number of coins remaining in the chest before the k^{th} pirate takes a share is $\frac{12}{12-k}$ times the number remaining afterward. Thus if there are n coins left for the 12^{th} pirate to take, the number of coins originally in the chest is

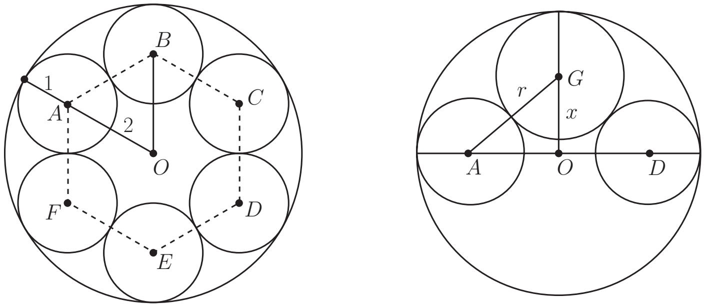
$$\frac{12^{11} \cdot n}{11!} = \frac{2^{22} \cdot 3^{11} \cdot n}{2^8 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11} = \frac{2^{14} \cdot 3^7 \cdot n}{5^2 \cdot 7 \cdot 11}.$$

The smallest value of n for which this is a positive integer is $5^2 \cdot 7 \cdot 11 = 1925$. In this case there are

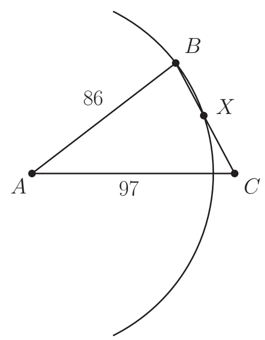
$$2^{14} \cdot 3^7 \cdot \frac{11!}{(12-k)! \cdot 12^{k-1}}$$

coins left for the k^{th} pirate to take, and note that this amount is an integer for each k. Hence the 12^{th} pirate receives 1925 coins.

22. **Answer (B):** Let the vertices of the regular hexagon be labeled in order A, B, C, D, E, and F. Let O be the center of the hexagon, which is also the center of the largest sphere. Let the eighth sphere have center G and radius r. Because the centers of the six small spheres are each a distance 2 from O and the small spheres have radius 1, the radius of the largest sphere is 3. Because G is equidistant from A and D, the segments $\overline{GO}$ and $\overline{AO}$ are perpendicular. Let x be the distance from G to O. Then $x + r = 3$. The Pythagorean Theorem applied to $\triangle AOG$ gives $(r+1)^2 = 2^2 + x^2 = 4 + (3-r)^2$, which simplifies to $2r + 1 = 13 - 6r$, so $r = \frac{3}{2}$. Note that this shows that the eighth sphere is tangent to $\overline{AD}$ at O.



23. **Answer (D):** By the Power of a Point Theorem, $BC \cdot CX = AC^2 - r^2$ where $r = AB$ is the radius of the circle. Thus $BC \cdot CX = 97^2 - 86^2 = 2013$. Since $BC = BX + CX$ and CX are both integers, they are complementary factors of 2013. Note that $2013 = 3 \cdot 11 \cdot 61$, and $CX < BC < AB + AC = 183$. Thus the only possibility is $CX = 33$ and $BC = 61$.



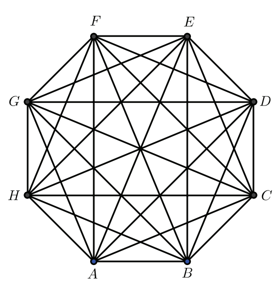
24. **Answer (E):** Call the players from Central A, B, and C, and call the players from Northern X, Y, and Z. Represent the schedule for each Central player by a string of length six consisting of two each of X, Y, and Z. There are $\binom{6}{2}\binom{4}{2} = 90$ possible strings for player A. Assume without loss of generality that the string is $XXYYZZ$. Player B's schedule must be a string with no X's in the first two positions, no Y's in the next two, and no Z's in the last two. If B's string begins with a Y and a Z in either order, the next two letters must be an X and a Z, and the last two must be an X and a Y. Because each pair can be ordered in one of two ways, there are $2^3 = 8$ such strings. If B's string begins with YY or ZZ, it must be $YYZZXX$ or $ZZXXYY$, respectively. Hence there are 10 possible schedules for B for each of the 90 schedules for A, and C's schedule is then determined. The total number of possible schedules is 900.

25. **Answer (A):** Label the octagon $ABCDEFGH$. There are 20 diagonals in all, 5 with endpoints at each vertex. The diagonals are of three types:

- Diagonals that skip over only one vertex, such as $\overline{AC}$ or $\overline{AG}$. These diagonals intersect with each of the five diagonals with endpoints at the skipped vertex.
- Diagonals that skip two vertices, such as $\overline{AD}$ or $\overline{AF}$. These diagonals intersect with four of the five diagonals that have endpoints at each of the two skipped vertices.
- Diagonals that cross to the opposite vertex, such as $\overline{AE}$. These diagonals intersect with three of the five diagonals that have endpoints at each of the three skipped vertices.

Therefore, from any given vertex, the diagonals will intersect other diagonals at $2\cdot5+2\cdot8+1\cdot9 = 35$ points. Counting from all 8 vertices, the total is $8\cdot35 = 280$ points.

Observe that, by symmetry, all four diagonals that cross to the opposite vertex intersect in the center of the octagon. This single intersection point has been counted 24 times, 3 from each of the 8 vertices. Further observe that at each of the vertices of the smallest internal octagon created by the diagonals, 3 diagonals intersect. For example, $\overline{AD}$ intersects with $\overline{CH}$ on $\overline{BF}$. These 8 intersection points have each been counted 12 times, 2 from each of the 6 affected vertices.The remaining intersection points each involve only two diagonals and each has been counted 4 times, once from each endpoint. These number $\frac{280-24-8\cdot12}{4} = 40$. There are therefore $1 + 8 + 40 = 49$ distinct intersection points in the interior of the octagon.



The problems and solutions in this contest were proposed by Betsy Bennett, George Brauer, Steve Blasberg, Steve Davis, Marta Eso, Josanne Furey, Michele Ghrist, Jerry Grossman, Peter Gilchrist, Jonathan Kane, Dan Kennedy, Joe Kennedy, Cap Khoury, Roy Roehl, Kevin Wang, Dave Wells, LeRoy Wenstrom, and Woody Wenstrom.

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