CHAPTER

9

TRIGONOMETRIC FUNCTIONS

A triangle is the most basic shape in our study of mathematics. The word **trigonometry** means the measurement of triangles. The study of the sun, Earth, and the other planets has been furthered by an understanding of the ratios of the sides of similar triangles. Eratosthenes (276–194 B.C.) used similar right triangles to estimate the circumference of Earth to about 25,000 miles. If we compare this to the best modern estimate, 24,902 miles, we see that even though his methods involved some inaccuracies, his final results were remarkable.

Although in the history of mathematics, the applications of trigonometry are based on the right triangle, the scope of trigonometry is much greater than that. Today, trigonometry is critical to fields ranging from computer art to satellite communications.

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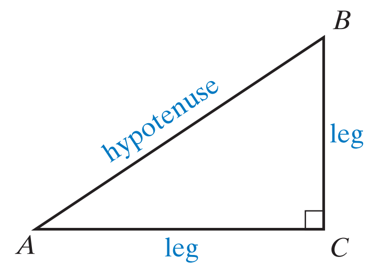
9-1 TRIGONOMETRY OF THE RIGHT TRIANGLE

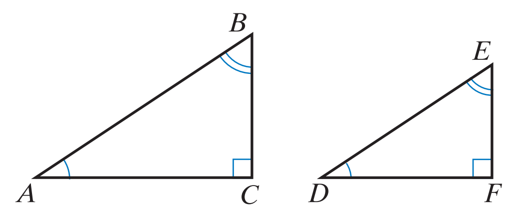
The right triangle is an important geometric figure that has many applications in the world around us. As we begin our study of trigonometry, we will recall what we already know about right triangles.

A right triangle has one right angle and two complementary acute angles. The side opposite the right angle is the longest side of the triangle and is called the **hypotenuse**. Each side opposite an acute angle is called a **leg**.

In the diagram, $\triangle ABC$ is a right triangle with $\angle C$ the right angle and $\overline{AB}$ the hypotenuse. The legs are $\overline{BC}$, which is opposite $\angle A$, and $\overline{AC}$, which is opposite $\angle B$. The leg $\overline{AC}$ is said to be adjacent to $\angle A$ and the leg $\overline{BC}$ is also said to be adjacent to $\angle B$. The acute angles, $\angle A$ and $\angle B$, are complementary.



When the corresponding angles of two triangles are congruent, the triangles are **similar**. In the diagram, $\triangle ABC$ and $\triangle DEF$ are right triangles with right angles $\angle C$ and $\angle F$. If $\angle A \cong \angle D$, then $\angle B \cong \angle E$ because complements of congruent angles are congruent. Therefore, the corresponding angles of $\triangle ABC$ and $\triangle DEF$ are congruent and $\triangle ABC \sim \triangle DEF$.



Two triangles are similar if and only if the lengths of their corresponding sides are proportional. For similar triangles $\triangle ABC$ and $\triangle DEF$, we can write:

$$\frac{BC}{EF} = \frac{AC}{DF} = \frac{AB}{DE}$$

In any proportion, we can interchange the means of the proportion to write a new proportion:

- Since $\frac{BC}{EF} = \frac{AB}{DE}$, we can write $\frac{BC}{AB} = \frac{EF}{DE}$.
- Since $\frac{AC}{DF} = \frac{AB}{DE}$, we can write $\frac{AC}{AB} = \frac{DF}{DE}$.
- Since $\frac{BC}{EF} = \frac{AC}{DF}$, we can write $\frac{BC}{AC} = \frac{EF}{DF}$.

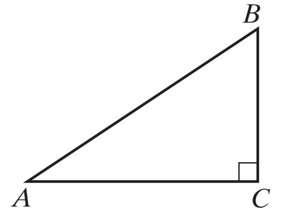
The ratio of any two sides of one triangle is equal to the ratio of the corresponding sides of a similar triangle.

Let $\triangle ABC$ be a right triangle with a right angle at C. Any other right triangle that has an acute angle congruent to $\angle A$ is similar to $\triangle ABC$. In $\triangle ABC$, $\overline{BC}$ is the leg that is opposite $\angle A$, $\overline{AC}$ is the leg that is adjacent to $\angle A$, and $\overline{AB}$ is the hypotenuse. We can write three ratios of sides of $\triangle ABC$. The value of each of these ratios is a function of the measure of $\angle A$, that is, a set of ordered pairs whose first element is the measure of $\angle A$ and whose second element is one and only one real number that is the value of the ratio.

$$\textbf{sine of } \angle A = \frac{BC}{AB} = \frac{\text{length of the leg opposite } \angle A}{\text{length of the hypotenuse}}$$

$$\textbf{cosine of } \angle A = \frac{AC}{AB} = \frac{\text{length of the leg adjacent to } \angle A}{\text{length of the hypotenuse}}$$

$$\textbf{tangent of } \angle A = \frac{BC}{AC} = \frac{\text{length of the leg opposite } \angle A}{\text{length of the leg adjacent to } \angle A}$$



We name each of these functions by using the first three letters of the name of the function just as we named the absolute value function by using the first three letters of the word "absolute." We also abbreviate the description of the sides of the triangle used in each ratio.

$$\sin A = \frac{\text{opp}}{\text{hyp}} \qquad \cos A = \frac{\text{adj}}{\text{hyp}} \qquad \tan A = \frac{\text{opp}}{\text{adj}}$$

Note: A word or sentence can be helpful in remembering these ratios. Use the first letter of the symbols in each ratio to make up a memory aid such as "SohCahToa," which comes from:

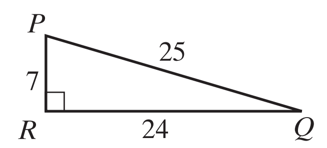
$$\text{Sin } A = \frac{\text{Opp}}{\text{Hyp}} \qquad \text{Cos } A = \frac{\text{Adj}}{\text{Hyp}} \qquad \text{Tan } A = \frac{\text{Opp}}{\text{Adj}}$$

EXAMPLE 1

In right triangle PQR, $\angle R$ is a right angle, $PQ = 25$, $QR = 24$, and $PR = 7$. Find the exact value of each trigonometric ratio.

a. $\sin P$ **b.** $\cos P$ **c.** $\tan P$ **d.** $\sin Q$ **e.** $\cos Q$ **f.** $\tan Q$

Solution The hypotenuse is $\overline{PQ}$ and $PQ = 25$.



The side opposite $\angle P$ and the side adjacent to $\angle Q$ is $\overline{QR}$ and $QR = 24$.

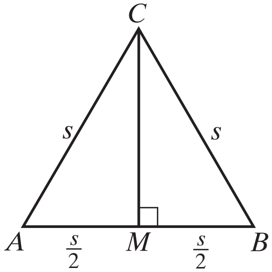
The side opposite $\angle Q$ and the side adjacent to $\angle P$ is $\overline{PR}$ and $PR = 7$.

a. $\sin P = \frac{\text{opp}}{\text{hyp}} = \frac{24}{25}$ **b.** $\cos P = \frac{\text{adj}}{\text{hyp}} = \frac{7}{25}$ **c.** $\tan P = \frac{\text{opp}}{\text{adj}} = \frac{24}{7}$

d. $\sin Q = \frac{\text{opp}}{\text{hyp}} = \frac{7}{25}$ **e.** $\cos Q = \frac{\text{adj}}{\text{hyp}} = \frac{24}{25}$ **f.** $\tan Q = \frac{\text{opp}}{\text{adj}} = \frac{7}{24}$

EXAMPLE 2

An altitude to a side of an equilateral triangle bisects that side forming two congruent right triangles. One of the acute angles of each right triangle is an angle of the equilateral triangle and has a measure of 60°. Show that $\sin 60° = \frac{\sqrt{3}}{2}$, $\cos 60° = \frac{1}{2}$, and $\tan 60° = \sqrt{3}$.



Solution Let s be the length of the equal sides, $\overline{AC}$ and $\overline{CB}$, of equilateral triangle ABC and $\overline{CM}$ the altitude to $\overline{AB}$. The midpoint of $\overline{AB}$ is M and $AM = \frac{s}{2}$.

Use the Pythagorean Theorem to find CM:

$$\begin{aligned}(AC)^2 &= (AM)^2 + (CM)^2\\ s^2 &= \left(\tfrac{s}{2}\right)^2 + (CM)^2\\ \tfrac{4s^2}{4} - \tfrac{s^2}{4} &= (CM)^2\\ \tfrac{3s^2}{4} &= (CM)^2\\ \tfrac{s\sqrt{3}}{2} &= CM\end{aligned}$$

Use the ratio of sides to find sine, cosine, and tangent:

The measure of $\angle A$ is 60, hyp $= AC = s$, opp $= CM = \frac{s\sqrt{3}}{2}$, and adj $= AM = \frac{s}{2}$.

$$\sin 60° = \frac{\text{opp}}{\text{hyp}} = \frac{CM}{AC} = \frac{\frac{s\sqrt{3}}{2}}{s} = \frac{\sqrt{3}}{2}$$

$$\cos 60° = \frac{\text{adj}}{\text{hyp}} = \frac{AM}{AC} = \frac{\frac{s}{2}}{s} = \frac{1}{2}$$

$$\tan 60° = \frac{\text{opp}}{\text{adj}} = \frac{CM}{AM} = \frac{\frac{s\sqrt{3}}{2}}{\frac{s}{2}} = \sqrt{3}$$

Note that the sine, cosine, and tangent values are independent of the value of s and therefore independent of the lengths of the sides of the triangle.

Exercises

Writing About Mathematics

1. In any right triangle, the acute angles are complementary. What is the relationship between the sine of the measure of an angle and the cosine of the measure of the complement of that angle? Justify your answer.

2. Bebe said that if A is the measure of an acute angle of a right triangle, $0 < \sin A < 1$. Do you agree with Bebe? Justify your answer.

Developing Skills

In 3–10, the lengths of the sides of $\triangle ABC$ are given. For each triangle, $\angle C$ is the right angle and $m\angle A < m\angle B$. Find: **a.** sin A **b.** cos A **c.** tan A.

3. 6, 8, 10 **4.** 5, 12, 13 **5.** 11, 60, 61 **6.** 8, 17, 15

7. 16, 30, 34 **8.** $2\sqrt{5}$, 2, 4 **9.** $\sqrt{2}$, 3, $\sqrt{7}$ **10.** 6, $3\sqrt{5}$, 9

11. Two of the answers to Exercises 3–10 are the same. What is the relationship between the triangles described in these two exercises? Justify your answer.

12. Use an isosceles right triangle with legs of length 3 to find the exact values of sin 45°, cos 45°, and tan 45°.

13. Use an equilateral triangle with sides of length 4 to find the exact values of sin 30°, cos 30°, and tan 30°.

Applying Skills

14. A 20-foot ladder leaning against a vertical wall reaches to a height of 16 feet. Find the sine, cosine, and tangent values of the angle that the ladder makes with the ground.

15. An access ramp reaches a doorway that is 2.5 feet above the ground. If the ramp is 10 feet long, what is the sine of the angle that the ramp makes with the ground?

16. The bed of a truck is 5 feet above the ground. The driver of the truck uses a ramp 13 feet long to load and unload the truck. Find the sine, cosine, and tangent values of the angle that the ramp makes with the ground.

17. A 20-meter line is used to keep a weather balloon in place. The sine of the angle that the line makes with the ground is $\frac{3}{4}$. How high is the balloon in the air?

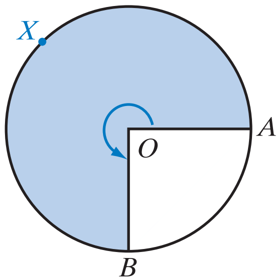
18. From a point on the ground that is 100 feet from the base of a building, the tangent of the angle of elevation of the top of the building is $\frac{5}{4}$. To the nearest foot, how tall is the building?

19. From the top of a lighthouse 75 feet high, the cosine of the angle of depression of a boat out at sea is $\frac{4}{5}$. To the nearest foot, how far is the boat from the base of the lighthouse?

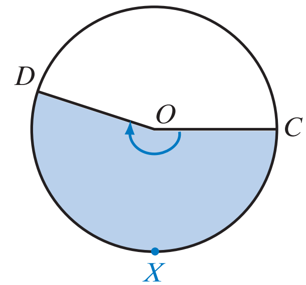
9-2 ANGLES AND ARCS AS ROTATIONS

Many machines operate by turning a wheel. As a wheel turns, each spoke from the center of the wheel moves through a central angle and any point on the rim of the wheel moves through an arc whose degree measure is equal to that of the central angle. The angle and the arc can have any real number as their measure.

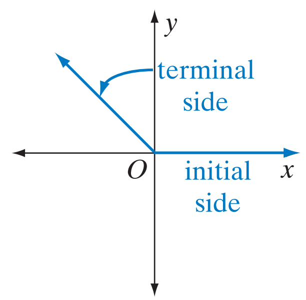
For example, the diagram shows a rotation of three right angles or 270° as point A moves to point B. The rotation is counterclockwise (opposite to the direction in which the hands of a clock move). A counterclockwise rotation is said to be a positive rotation. Therefore, $m\angle AOB = m\widehat{AXB} = 270$.



When a wheel turns in a clockwise direction (the same direction in which the hands of a clock turn), the rotation is said to be negative. For example, the diagram shows a rotation of $-200°$ as point C moves to point D. Therefore, $m\angle COD = m\widehat{CXD} = -200$. The ray at which an angle of rotation begins is the **initial side** of the angle. Here, $\overrightarrow{OC}$ is the initial side of $\angle COD$. The ray at which an angle of rotation ends is the **terminal side** of the angle. Here, $\overrightarrow{OD}$ is the terminal side of $\angle COD$.

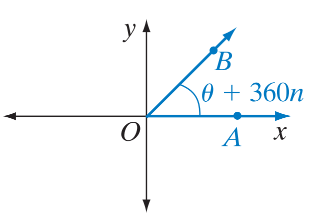


In order to study rotations through angles and arcs, and the lengths of line segments associated with these angles and arcs, we compare the rotations with central angles and arcs of a circle in the coordinate plane. An angle is in **standard position** when its vertex is at the origin and its initial side is the nonnegative ray of the x-axis.

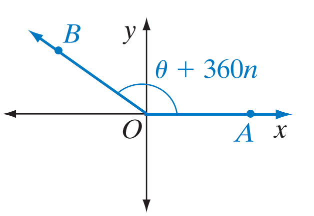


We classify angles in standard position according to the quadrant in which the terminal side lies. The Greek symbol $\boldsymbol{\theta}$ (theta) is often used to represent angle measure.

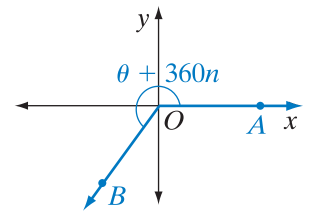
- If $0 < \theta < 90$ and $m\angle AOB = \theta + 360n$ for any integer n, then $\angle AOB$ is a first-quadrant angle.



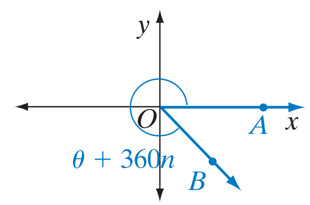
- If $90 < \theta < 180$ and $m\angle AOB = \theta + 360n$ for any integer n, then $\angle AOB$ is a second-quadrant angle.



- If $180 < \theta < 270$ and $m\angle AOB = \theta + 360n$ for any integer n, then $\angle AOB$ is a third-quadrant angle.



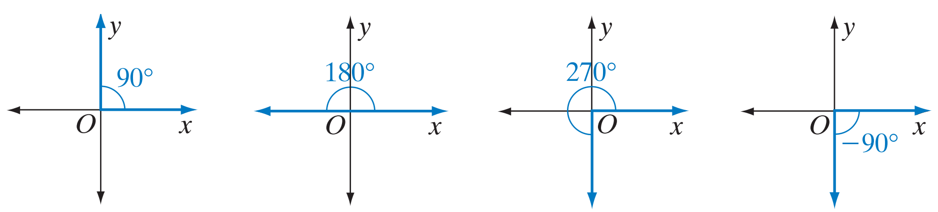
- If $270 < \theta < 360$ and $m\angle AOB = \theta + 360n$ for any integer n, then $\angle AOB$ is a fourth-quadrant angle.



An angle in standard position whose terminal side lies on either the x-axis or the y-axis is called a **quadrantal angle**.

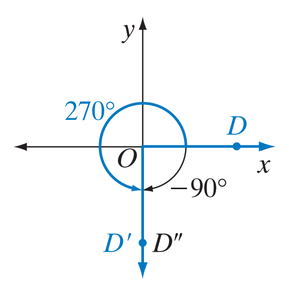
► **The degree measure of a quadrantal angle is a multiple of 90.**

The following figures show examples of quadrantal angles:



Coterminal Angles

When D on the positive ray of the x-axis is rotated counterclockwise about the origin 270° to point D', a quadrantal angle in standard position, $\angle DOD'$, is formed. When point D on the positive ray of the x-axis is rotated clockwise 90° to point D'', $\overrightarrow{OD'}$ and $\overrightarrow{OD''}$ coincide. We say that $\angle DOD''$ and $\angle DOD'$ are *coterminal angles* because they have the same terminal side. The measure of $\angle DOD'$ is 270° and the measure of $\angle DOD''$ is −90°. The measure of $\angle DOD''$ and the measure of $\angle DOD'$ differ by 360° or a complete rotation. The measure of $\angle DOD''$, −90, can be written as $270 + 360(-1)$ or $m\angle DOD'$ plus a multiple of 360.



DEFINITION

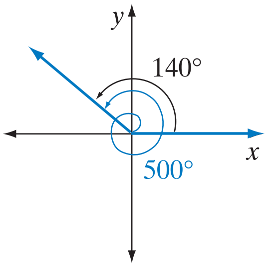
Angles in standard position that have the same terminal side are **coterminal angles**.

If $\angle AOB$ and $\angle AOC$ are angles in standard position with the same terminal side, then $m\angle AOB = m\angle AOC + 360n$ for some integer n.

EXAMPLE 1

An angle in standard form with a measure of 500° lies in what quadrant?

Solution Find a coterminal angle of the 500° angle with a measure between 0° and 360°.



The measures of coterminal angles can be found by adding or subtracting multiples of 360°, the measure of a complete rotation.

$$500 - 360 = 140$$

Therefore, angles of 500° and 140° are coterminal.

An angle of 140° is in quadrant II.

An angle of 500° is a second-quadrant angle. *Answer*

EXAMPLE 2

Find the measures of five angles that are coterminal with an angle of 120°.

Solution If angles are coterminal, then their degree measures differ by a multiple of 360°. The measures of angles coterminal with an angle of 120° are of the form $120 + 360n$ for any integer n. Five of the infinitely many coterminal angle measures are given below:

$$120 + 360(1) = 480 \qquad 120 + 360(2) = 840 \qquad 120 + 360(3) = 1{,}200$$

$$120 + 360(-1) = -240 \qquad 120 + 360(-2) = -600$$

Angles whose measures are −600°, −240°, 480°, 840°, and 1,200° are all coterminal with an angle of 120°. *Answer*

Exercises

Writing About Mathematics

1. Is an angle of 810° a quadrantal angle? Explain why or why not.
2. Huey said that if the sum of the measures of two angles in standard position is a multiple of 360, then the angles are coterminal. Do you agree with Huey? Explain why or why not.

Developing Skills

In 3–7, draw each angle in standard position.

3. 45° **4.** 540° **5.** −180° **6.** −120° **7.** 110°

In 8–17, name the quadrant in which an angle of each given measure lies.

8. 25° **9.** 150° **10.** 200° **11.** 300° **12.** −75°

13. −200° **14.** −280° **15.** −400° **16.** 750° **17.** 1,050°

In 18–27, for each given angle, find a coterminal angle with a measure of θ such that $0 \leq \theta < 360$.

18. 390° **19.** 412° **20.** 1,000° **21.** −10° **22.** −85°

23. –270° **24.** −500° **25.** 540° **26.** 360° **27.** 980°

Applying Skills

28. Do the wheels of a car move in the clockwise or counterclockwise direction when the car is moving to the right of a person standing at the side of the car?

29. To remove the lid of a jar, should the lid be turned clockwise or counterclockwise?

30. a. To insert a screw, should the screw be turned clockwise or counterclockwise?

b. The thread spirals six and half times around a certain screw. How many degrees should the screw be turned to insert it completely?

31. The blades of a windmill make one complete rotation per second. How many rotations do they make in one minute?

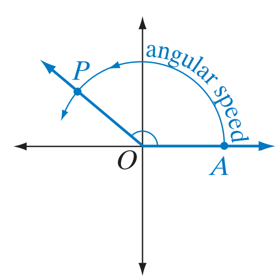
32. An airplane propeller rotates 750 times per minute. How many times will a point on the edge of the propeller rotate in 1 second?

33. The Ferris wheel at the county fair takes 2 minutes to complete one full rotation.

a. To the nearest second, how long does it take the wheel to rotate through an angle of 260°?

b. How many minutes will it take for the wheel to rotate through an angle of 1,125°?

34. The measure of angle POA changes as P is rotated around the origin. The ratio of the change in the measure of the angle to the time it takes for the measure to change is called the **angular speed** of point P. For example, if a ceiling fan rotates 30 times per minute, its angular speed in degrees per minute is:



$$30(360°) = 10{,}800° \text{ per minute}$$

Find the angular speed in degrees per second of a tire rotating:

a. 3 times per minute

b. 90 times per minute

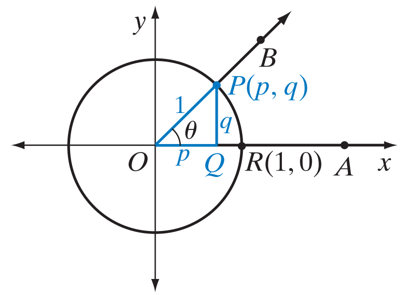
c. 600 times per minute

9-3 THE UNIT CIRCLE, SINE, AND COSINE

Wheels, machinery, and other items that rotate suggest that trigonometry must apply to angles that have measures greater than a right angle and that have negative as well as positive measures. We want to write definitions of sine, cosine, and tangent that include the definitions that we already know but go beyond them. We will use a circle in the coordinate plane to do this.

A circle with center at the origin and radius 1 is called the **unit circle** and has the equation $x^2 + y^2 = 1$. Let θ be the measure of a central angle in standard position.

Let $\angle AOB$ be an angle in standard position with its terminal side in the first quadrant and $m\angle AOB = \theta$. Let $P(p, q)$ be the point at which the terminal side intersects the unit circle and Q be the point at which the vertical line through P intersects the x-axis. Triangle POQ is a right triangle with right angle at Q, $m\angle QOP = m\angle AOB = \theta$, $PQ = q$, $OQ = p$, and $OP = 1$.



$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{OQ}{OP} = \frac{p}{1} = p$$

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{PQ}{OP} = \frac{q}{1} = q$$

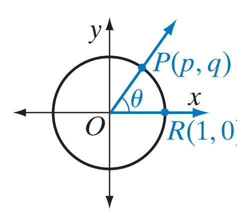
Therefore, the coordinates of P are $(\cos\theta, \sin\theta)$. We have shown that for a first-quadrant angle in standard position, the cosine of the angle measure is the x-coordinate of the point at which the terminal side of the angle intersects the unit circle and the sine of the angle measure is the y-coordinate of the same point. Since P is in the first quadrant, p and q are positive numbers and $\cos\theta$ and $\sin\theta$ are positive numbers.

We can use the relationship between the point P on the unit circle and angle POR in standard position to define the sine and cosine functions for *any* angle, not just first-quadrant angles. In particular, notice that $\angle POR$ with measure θ determines the y-coordinate of P for any value of θ. This y-value is defined to be $\sin\theta$. Similarly, $\angle POR$ with measure θ determines the x-coordinate of P for any value of θ. This x-value is defined to be $\cos\theta$.

Let us consider angle $\angle POR$ in each quadrant.

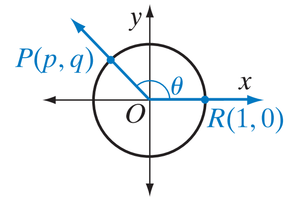
CASE I *First-quadrant angles*

$\angle POR$ is a first-quadrant angle.
The coordinates of P are (p, q).
$\cos\theta = p$ is positive.
$\sin\theta = q$ is positive.



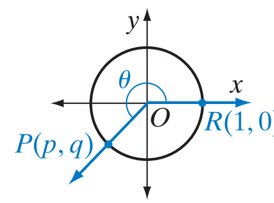
CASE 2 *Second-quadrant angles*

$\angle POR$ is a second-quadrant angle.
The coordinates of P are (p, q).
$\cos\theta = p$ is negative.
$\sin\theta = q$ is positive.



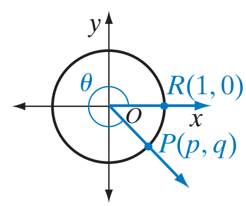
CASE 3 *Third-quadrant angles*

$\angle POR$ is a third-quadrant angle.
The coordinates of P are (p, q).
$\cos\theta = p$ is negative.
$\sin\theta = q$ is negative.

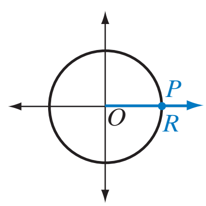


CASE 4 *Fourth-quadrant angles*

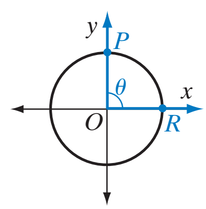
$\angle POR$ is a fourth-quadrant angle.
The coordinates of P are (p, q).
$\cos\theta = p$ is positive.
$\sin\theta = q$ is negative.



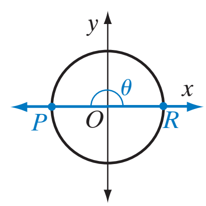
CASE 5 *Quadrantal angles*



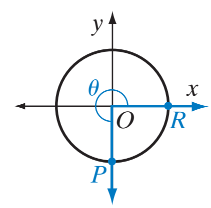
$\theta = 0°$
$\cos\theta = 1$
$\sin\theta = 0$



$\theta = 90°$
$\cos\theta = 0$
$\sin\theta = 1$



$\theta = 180°$
$\cos\theta = -1$
$\sin\theta = 0$



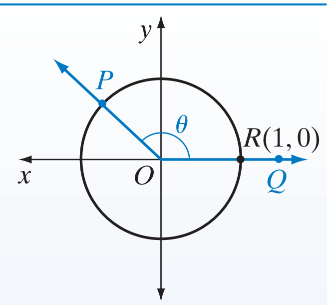
$\theta = 270°$
$\cos\theta = 0$
$\sin\theta = -1$

These examples allow us to use the unit circle to define the sine and cosine functions for the measure of any angle or rotation.

DEFINITION

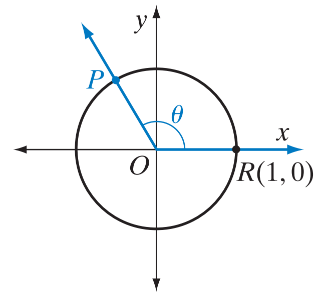
Let $\angle POQ$ be an angle in standard position and P be the point where the terminal side of the angle intersects the unit circle. Let $m\angle POQ = \theta$. Then:

- The **sine function** is the set of ordered pairs $(\theta, \sin\theta)$ such that $\sin\theta$ is the y-coordinate of P.
- The **cosine function** is the set of ordered pairs $(\theta, \cos\theta)$ such that $\cos\theta$ is the x-coordinate of P.



EXAMPLE 1

If $P\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ is a point on the unit circle and on the terminal side of an angle in standard position whose measure is θ, find: **a.** $\sin\theta$ **b.** $\cos\theta$

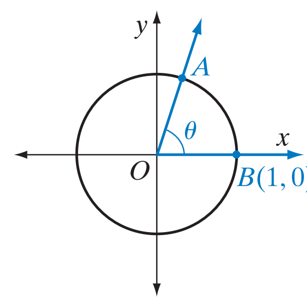


Solution **a.** $\sin\theta = y\text{-coordinate of } P = \frac{\sqrt{3}}{2}$. *Answer*

b. $\cos\theta = x\text{-coordinate of } P = -\frac{1}{2}$. *Answer*

EXAMPLE 2

Angle AOB is an angle in standard position with measure θ. If $\theta = 72°$ and A is a point on the unit circle and on the terminal side of $\angle AOB$, find the coordinates of A to the nearest hundredth.

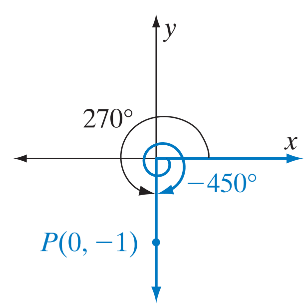


Solution The coordinates of $A = (\cos 72°, \sin 72°)$.

Since $\cos 72° \approx 0.3091$ and $\sin 72° \approx 0.9515$, the coordinates of A to the nearest hundredth are $(0.31, 0.95)$. *Answer*

EXAMPLE 3

Find $\sin(-450°)$ and $\cos(-450°)$.



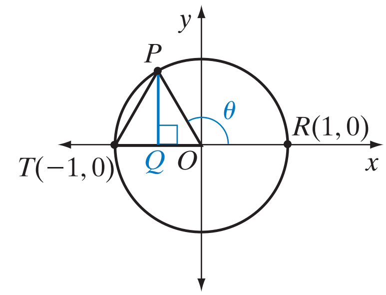
Solution An angle in standard position whose measure is -450 has the same terminal side as an angle of $-450 + 2(360)$ or $270°$. Let P be the point where the terminal side of an angle of $270°$ intersects the unit circle. The coordinates of P are $(0, -1)$.

$$\sin(-450°) = \sin 270° = -1$$
$$\cos(-450°) = \cos(-450°) = 0$$

Therefore, $\sin(-450°) = -1$ and $\cos(-450°) = 0$. *Answer*

EXAMPLE 4

In the diagram $\triangle OPT$ is an equilateral triangle with O at the origin, P on the unit circle, $T(-1, 0)$, and $R(1, 0)$. Let $\theta = m\angle ROP$. Find: **a.** θ **b.** $\cos\theta$ **c.** $\sin\theta$



Solution **a.** Since $\angle ROP$ and $\angle TOP$ are a linear pair of angles, they are supplementary. The measure of an angle of an equilateral triangle is 60°. Therefore, $m\angle TOP = 60$ and $m\angle ROP = 120$ or $\theta = 120$.

b. The value of $\cos\theta$ or $\cos 120$ is equal to the x-coordinate of P. Draw $\overline{PQ}$, the altitude from P in $\triangle OPT$. In an equilateral triangle, the altitude bisects the side to which it is drawn. Since $OT = 1$, $OQ = \frac{1}{2}$. The x-coordinate of P is negative and equal to $-OQ$. Therefore, $\cos\theta = -OQ = -\frac{1}{2}$.

c. The value of $\sin\theta$ or $\sin 120$ is equal to the y-coordinate of P. Triangle OQP is a right triangle with $OP = 1$.

$$\begin{aligned}
OP^2 &= OQ^2 + PQ^2 \\
1^2 &= \left(\tfrac{1}{2}\right)^2 + PQ^2 \\
1 &= \tfrac{1}{4} + PQ^2 \\
\tfrac{4}{4} - \tfrac{1}{4} &= PQ^2 \\
\tfrac{3}{4} &= PQ^2 \\
\tfrac{\sqrt{3}}{2} &= PQ
\end{aligned}$$

The y-coordinate of P is positive and equal to PQ. Therefore, $\sin\theta = PQ = \frac{\sqrt{3}}{2}$.

Answers **a.** 120° **b.** $\cos 120° = -\frac{1}{2}$ **c.** $\sin 120° = \frac{\sqrt{3}}{2}$

The values given here are exact values. Compare these values with the rational approximations given on a calculator. Press **MODE** on your calculator. In the third line, DEGREE should be highlighted.

ENTER: **SIN** 120 **)** **ENTER** **2nd** **√** 3 **)** **÷** 2 **ENTER**

ENTER: **COS** 120 **)** **ENTER** −1 **÷** 2 **ENTER**

DISPLAY:

```
sin(120)
            .8660254038
√(3)/2
            .8660254038
```

DISPLAY:

```
cos(120)
                     -.5
-1/2
                     -.5
```

Exercises

Writing About Mathematics

1. If P is the point at which the terminal side of an angle in standard position intersects the unit circle, what are the largest and smallest values of the coordinates of P? Justify your answer.

2. Are the sine function and the cosine function one-to-one functions? Justify your answer.

Developing Skills

In 3–10, the terminal side of $\angle ROP$ in standard position intersects the unit circle at P. If $m\angle ROP$ is θ, find: **a.** $\sin\theta$ **b.** $\cos\theta$ **c.** the quadrant of $\angle ROP$

3. $P\left(\frac{3}{5}, \frac{4}{5}\right)$ **4.** $P(0.6, -0.8)$ **5.** $P\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ **6.** $P\left(\frac{\sqrt{5}}{5}, -\frac{2\sqrt{5}}{5}\right)$

7. $P\left(-\frac{5}{13}, -\frac{12}{13}\right)$ **8.** $P\left(\frac{24}{25}, -\frac{7}{25}\right)$ **9.** $P\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ **10.** $P\left(-\frac{9}{41}, \frac{40}{41}\right)$

In 11–14, for each of the following function values, find θ if $0° \leq \theta < 360°$.

11. $\sin\theta = 1$, $\cos\theta = 0$ **12.** $\sin\theta = -1$, $\cos\theta = 0$ **13.** $\sin\theta = 0$, $\cos\theta = 1$ **14.** $\sin\theta = 0$, $\cos\theta = -1$

In 15–22, for each given angle in standard position, determine to the nearest tenth the coordinates of the point where the terminal side intersects the unit circle.

15. 90° **16.** −180° **17.** 81° **18.** 137°

19. 229° **20.** 312° **21.** 540° **22.** −45°

23. If $P\left(\frac{2}{3}, y\right)$ is a point on the unit circle and on the terminal side of an angle in standard position with measure θ, find: **a.** y **b.** $\sin\theta$ **c.** $\cos\theta$

Applying Skills

24. Let $\angle ROP$ be an angle in standard position, P the point at which the terminal side intersects the unit circle and $m\angle ROP = \theta$.

a. Under the dilation D_a, the image of $A(x, y)$ is $A'(ax, ay)$. What are the coordinates P' of the image of P under the dilation D_5?

b. What are the coordinates of P'', the image of P under the dilation D_{-5}?

c. Express $m\angle ROP'$ and $m\angle ROP''$ in terms of θ.

25. Under a reflection in the y-axis, the image of $A(x, y)$ is $A'(-x, y)$. The measure of $\angle ROP = \theta$ and $P(\cos\theta, \sin\theta)$ is a point on the terminal side of $\angle ROP$. Let P' be the image of P and R' be the image of R under a reflection in the y-axis.

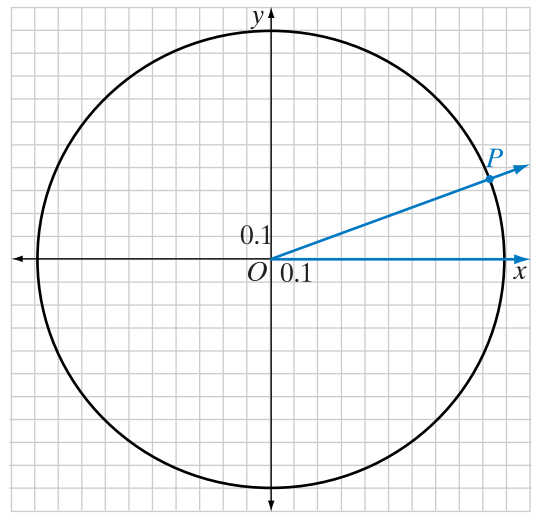
a. What are the coordinates of P'?

b. Express the measure of $\angle R'OP'$ in terms of θ.

c. Express the measure of $\angle ROP'$ in terms of θ.

Hands-On Activity

Use graph paper, a protractor, a ruler, and a pencil, or a computer program for this activity.



1. Let the side of each square of the grid represent 0.1. Draw a circle with center at the origin and radius 1.
2. Draw the terminal side of an angle of 20° in standard position. Label the point where the terminal side intersects the circle *P*.
3. Estimate the coordinates of *P* to the nearest hundredth.
4. Use a calculator to find sin 20° and cos 20°.
5. Compare the numbers obtained from the calculator with the coordinates of *P*.
6. Repeat steps 2 through 5, replacing 20° with the following values: 70°, 100°, 165°, 200°, 250°, 300°, 345°.

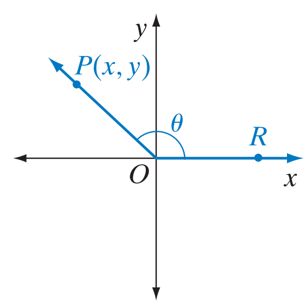
In each case, are the values of the sine and the cosine approximately equal to the coordinates of *P*?

Hands-On Activity: Finding Sine and Cosine Using *Any* Point on the Plane

For any $\angle ROP$ in standard position, point $P(x, y)$ is on the terminal side, $OP = r = \sqrt{x^2 + y^2}$, and $m\angle ROP = \theta$. The trigonometric function values of θ are:

$$\sin \theta = \frac{y}{r} \qquad \cos \theta = \frac{x}{r}$$

Note that the point *P* is no longer restricted to the unit circle.



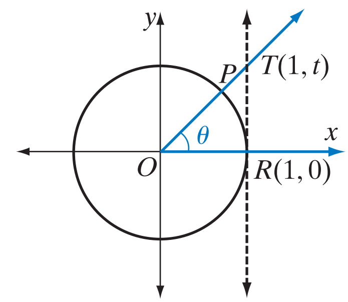
In (1)–(8), the given point is on the terminal side of angle θ. Find r, $\sin \theta$, and $\cos \theta$.

(1) $(3, 4)$ (2) $(-5, 12)$ (3) $(8, -15)$ (4) $(-2, -7)$

(5) $(1, 7)$ (6) $(-1, 7)$ (7) $(1, -2)$ (8) $(-3, -3)$

9-4 THE TANGENT FUNCTION

We have used the unit circle and an angle in standard position to write a general definition for the sine and cosine functions of an angle of any measure. We can write a similar definition for the tangent function in terms of a line *tangent* to the unit circle.



We will begin with an angle in the first quadrant. Angle ROP is an angle in standard position and P is the point at which the terminal side of $\angle ROP$ intersects the unit circle. Let $m\angle ROP = \theta$. At R, the point at which the unit circle intersects the positive ray of the x-axis, construct a line tangent to the circle and let T be the point at which the terminal side intersects this tangent line. Recall that a line tangent to the circle is perpendicular to the radius drawn to the point at which the tangent line intersects the circle. Since $\overline{OR}$ is a portion of the x-axis, $\overline{RT}$ is a vertical line. The x-coordinate of T is the same as the x-coordinate of R. The coordinates of T are $(1, t)$. Since $\angle ORT$ is a right angle and $\triangle ORT$ is a right triangle, $OR = 1$ and $RT = t$.

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{RT}{OR} = \frac{t}{1} = t$$

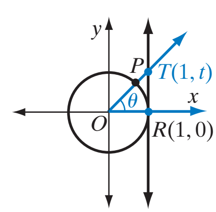
Therefore, the coordinates of T are $(1, \tan\theta)$. We have shown that for a first-quadrant angle in standard position with measure θ, $\tan\theta$ is equal to the y-coordinate of the point where the terminal side of the angle intersects the line that is tangent to the unit circle at the point $(1, 0)$. Since T is in the first quadrant, t is a positive number and $\tan\theta$ is a positive number.

We can use this relationship between the point T and the tangent to the unit circle to define the tangent function for angles other than first-quadrant angles. In particular, $\angle ROP$ with measure θ determines the y-coordinate of T for any value of θ. This y-value is defined to be $\tan\theta$.

Let us consider angle $\angle ROP$ in each quadrant.

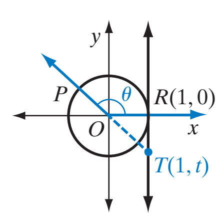
CASE 1 *First-quadrant angles*

$\angle ROP$ is a first-quadrant angle.
The coordinates of T are $(1, t)$.
$\tan\theta = t$ is positive.



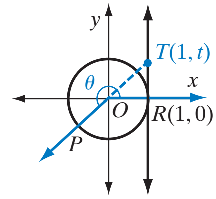
CASE 2 *Second-quadrant angles*

$\angle ROP$ is a second-quadrant angle.
Extend the terminal side through O.
The coordinates of T are $(1, t)$.
$\tan\theta = t$ is negative.



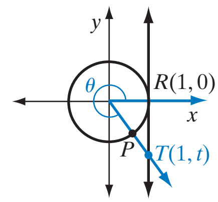
CASE 3 *Third-quadrant angles*

$\angle ROP$ is a third-quadrant angle.
Extend the terminal side through O.
The coordinates of T are $(1, t)$.
$\tan\theta = t$ is positive.

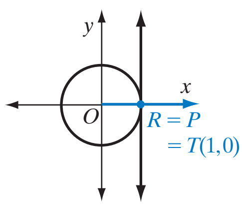


CASE 4 *Fourth-quadrant angles*

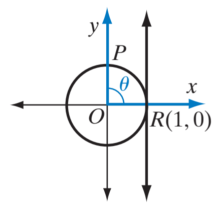
$\angle ROP$ is a fourth-quadrant angle.
The coordinates of T are $(1, t)$.
$\tan\theta = t$ is negative.



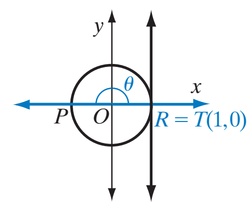
CASE 5 *Quadrantal angles*



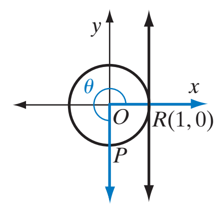
$\theta = 0°$
$\tan 0° = 0$



$\theta = 90°$
no intersection
$\tan 90°$ is undefined



$\theta = 180°$
$\tan 180° = 0$



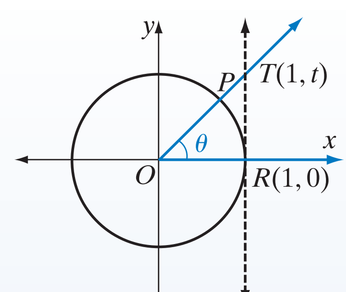
$\theta = 270°$
no intersection
$\tan 270°$ is undefined

We have extended the definition of the tangent function on the unit circle for any angle or rotation for which $\tan\theta$ is defined and to determine those angles or rotations for which $\tan\theta$ is not defined.

DEFINITION

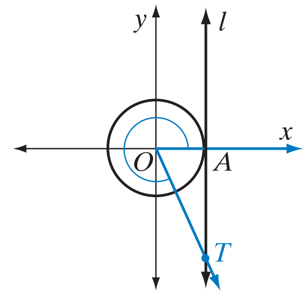
Let T be the point where the terminal side of an angle in standard position intersects the line that is tangent to the unit circle at $R(1, 0)$. Let $m\angle ROP = \theta$. Then:

- The **tangent function** is a set of ordered pairs $(\theta, \tan\theta)$ such that $\tan\theta$ is the y-coordinate of T.



EXAMPLE 1

Line l is tangent to the unit circle at $A(1, 0)$. If $T(1, -\sqrt{5})$ is a point on l, what is $\tan \angle AOT$?



Solution $\tan \angle AOT = y\text{-coordinate of } T$
$= -\sqrt{5}$ *Answer*

EXAMPLE 2

Angle AOT is an angle in standard position with measure θ. If $\theta = 242°$ and T is a point on the line tangent to the unit circle at $A(1, 0)$, find the coordinates of T to the nearest hundredth.

Solution The coordinates of $T = (1, \tan 242°)$.

Since $\tan 242° \approx 1.8807$, the coordinates of T to the nearest hundredth are $(1, 1.88)$. *Answer*

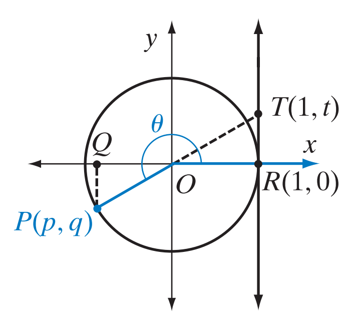
EXAMPLE 3

Angle AOB is an angle in standard position and $m\angle AOB = \theta$. If $\sin \theta < 0$ and $\tan \theta < 0$, in what quadrant is $\angle AOB$?

Solution When the sine of an angle is negative, the angle could be in quadrants III or IV. When the tangent of an angle is negative, the angle could be in quadrants II or IV. Therefore, when both the sine and the tangent of the angle are negative, the angle must be in quadrant IV. *Answer*

EXAMPLE 4

In the diagram, $\overleftrightarrow{RT}$ is tangent to the unit circle at $R(1, 0)$. The terminal ray of $\angle ROP$ intersects the unit circle at $P(p, q)$ and the line that contains the terminal ray of $\angle ROP$ intersects the tangent line at $T(1, t)$. The vertical line from P intersects the x-axis at $Q(p, 0)$.



a. Prove that if $m\angle ROP = \theta$, then $\tan \theta = \frac{\sin \theta}{\cos \theta}$.

b. If the coordinates of P are $\left(-\frac{2\sqrt{2}}{3}, -\frac{1}{3}\right)$, find $\sin \theta$, $\cos \theta$, and $\tan \theta$.

Solution **a.** In the diagram, $\overline{RT}$ and $\overline{PQ}$ are vertical segments perpendicular to the x-axis. Therefore, $\triangle ROT$ and $\triangle QOP$ are right triangles. The vertical angles $\angle ROT$ and $\angle QOP$ are congruent. Therefore, $\triangle ROT \sim \triangle QOP$ by AA$\sim$. The ratios of corresponding sides of similar triangles are in proportion.

(1) Write a proportion using the lengths of the legs of the right triangles. The lengths of the legs are positive numbers. $RT = |t|$, $QP = |q|$, and $OQ = |p|$:

$$\frac{RT}{OR} = \frac{QP}{OQ}$$

$$\frac{|t|}{1} = \frac{|q|}{|p|}$$

(2) Since t is a positive number, $|t| = t$. Since p and q are negative numbers, $|p| = -p$ and $|q| = -q$:

$$\frac{t}{1} = \frac{-q}{-p}$$

$$t = \frac{q}{p}$$

(3) Replace t, q, and p with the function values that they represent:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

b. The coordinates of P are $\left(-\frac{2\sqrt{2}}{3}, -\frac{1}{3}\right)$, $\cos\theta = -\frac{2\sqrt{2}}{3}$, $\sin\theta = -\frac{1}{3}$, and:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{1}{3}}{-\frac{2\sqrt{2}}{3}} = \frac{1}{2\sqrt{2}} = \frac{1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{4} \quad \textit{Answer}$$

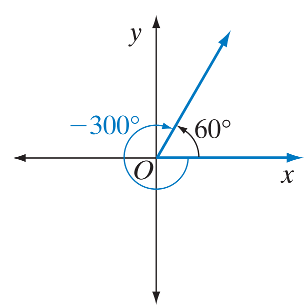
What we have established in the last example is that for a given value of θ, the tangent function value is equal to the ratio of the sine function value to the cosine function value.

$$\boldsymbol{\tan\theta = \frac{\sin\theta}{\cos\theta}}$$

EXAMPLE 5

Use the properties of coterminal angles to find tan $(-300°)$.

Solution Coterminal angles are angles in standard position that have the same terminal side.



$$-300 + 360 = 60$$

Therefore, a $-300°$ angle is coterminal with a $60°$ angle.

$$\sin(-300°) = \sin 60° = \frac{\sqrt{3}}{2}$$

$$\cos(-300°) = \cos 60° = \frac{1}{2}$$

$$\tan(-300°) = \frac{\cos(-300°)}{\sin(-300°)} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3} \quad \textit{Answer}$$

Exercises

Writing About Mathematics

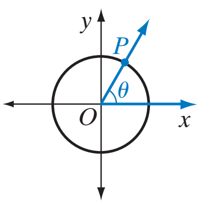
1. a. What are two possible measures of θ if $0° < \theta < 360°$ and $\sin\theta = \cos\theta$? Justify your answer.

   b. What are two possible measures of θ if $0° < \theta < 360°$ and $\tan\theta = 1$? Justify your answer.

2. What is the value of $\cos\theta$ when $\tan\theta$ is undefined? Justify your answer.

Developing Skills

In 3–11, P is the point at which the terminal side of an angle in standard position intersects the unit circle. The measure of the angle is θ. For each point P, the x-coordinate and the quadrant is given. Find: **a.** the y-coordinate of P **b.** $\cos\theta$ **c.** $\sin\theta$ **d.** $\tan\theta$



3. $\left(\frac{3}{5}, y\right)$, first quadrant

4. $\left(\frac{5}{13}, y\right)$, fourth quadrant

5. $\left(-\frac{6}{10}, y\right)$, second quadrant

6. $\left(-\frac{1}{4}, y\right)$, third quadrant

7. $\left(-\frac{\sqrt{3}}{2}, y\right)$, second quadrant

8. $\left(\frac{\sqrt{5}}{3}, y\right)$, first quadrant

9. $\left(-\frac{\sqrt{2}}{2}, y\right)$, third quadrant

10. $\left(\frac{1}{5}, y\right)$, fourth quadrant

11. $\left(\frac{3}{4}, y\right)$, first quadrant

12. $\left(-\frac{\sqrt{7}}{3}, y\right)$, second quadrant

In 13–20, P is a point on the terminal side of an angle in standard position with measure θ and on a circle with center at the origin and radius r. For each point P, find: **a.** r **b.** $\cos\theta$ **c.** $\sin\theta$ **d.** $\tan\theta$

13. $(7, 24)$ **14.** $(8, 15)$ **15.** $(-1, -1)$ **16.** $(-3, -4)$

17. $(-3, 6)$ **18.** $(-2, 6)$ **19.** $(4, -4)$ **20.** $(9, -3)$

In 21–26, if θ is the measure of $\angle AOB$, an angle in standard position, name the quadrant in which the terminal side of $\angle AOB$ lies.

21. $\sin\theta > 0, \cos\theta > 0$ **22.** $\sin\theta < 0, \cos\theta > 0$ **23.** $\sin\theta < 0, \cos\theta < 0$

24. $\sin\theta > 0, \tan\theta > 0$ **25.** $\tan\theta < 0, \cos\theta < 0$ **26.** $\sin\theta < 0, \tan\theta > 0$

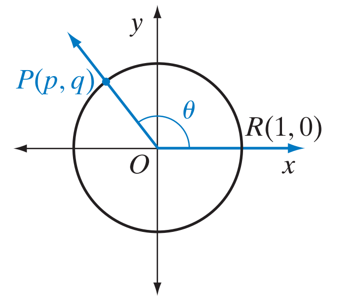
27. When $\sin\theta = -1$, find a value of: **a.** $\cos\theta$ **b.** $\tan\theta$ **c.** θ

28. When $\tan\theta = 0$, find a value of: **a.** $\sin\theta$ **b.** $\cos\theta$ **c.** θ

29. When $\tan\theta$ is undefined, find a value of: **a.** $\sin\theta$ **b.** $\cos\theta$ **c.** θ

Applying Skills

30. Angle ROP is an angle in standard position with $m\angle ROP = \theta$, $R(1, 0)$ a point on the initial side of $\angle ROP$, and $P(p, q)$ the point at which the terminal side of $\angle ROP$ intersects the unit circle.



a. What is the domain of cos θ and of sin θ?

b. What is the range of cos θ and of sin θ?

c. Is tan θ defined for all angle measures?

d. What is the domain of tan θ?

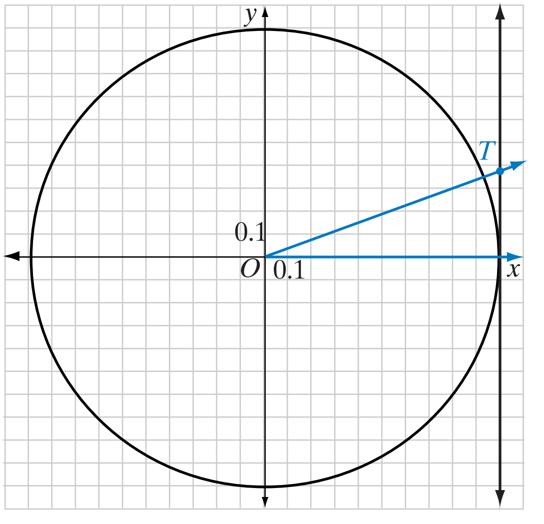
e. What is the range of tan θ?

31. Use the definitions of sin θ and cos θ based on the unit circle to prove that $\sin^2\theta + \cos^2\theta = 1$.

32. Show that if $\angle ROP$ is an angle in standard position and $m\angle ROP = \theta$, then the slope of $\overleftrightarrow{PO} = \tan\theta$.

Hands-On Activity

Use graph paper, protractor, ruler, and pencil, or geometry software for this activity.



1. On graph paper, let the side of each square represent 0.1. Draw a circle with center at the origin and radius 1 and the line tangent to the circle at $(1, 0)$.
2. Draw the terminal side of an angle of 20°. Label point T, the point at which the terminal side intersects the tangent.
3. Estimate the y-coordinate of T to the nearest hundredth.
4. Use a calculator to find tan 20.
5. Compare the number obtained from the calculator with the y-coordinates of T.
6. Repeat steps 2 through 5, replacing 20° with the following values: 70°, 100°, 165°, 200°, 250°, 300°, 345°.

In each case, are the values of the tangent approximately equal to the y-coordinate of T?

9-5 THE RECIPROCAL TRIGONOMETRIC FUNCTIONS

We have defined three trigonometric functions: sine, cosine, and tangent. There are three other trigonometric functions that can be defined in terms of sin θ, cos θ, and tan θ. These functions are the **reciprocal functions**.

The Secant Function

DEFINITION

The **secant function** is the set of ordered pairs $(\theta, \sec\theta)$ such that for all θ for which $\cos\theta \neq 0$, $\sec\theta = \frac{1}{\cos\theta}$.

We know that $\cos\theta = 0$ for $\theta = 90$, $\theta = 270$, and for any value of θ that differs from 90 or 270 by 360. Since $270 = 90 + 180$, $\cos\theta = 0$ for all values of θ such that $\theta = 90 + 180n$ where n is an integer. Therefore, $\sec\theta$ is defined for the degree measures of angles, θ, such that $\theta \neq 90 + 180n$.

The secant function values are the reciprocals of the cosine function values. We know that $-1 \leq \cos\theta \leq 1$. The reciprocal of a positive number less than or equal to 1 is a positive number greater than or equal to 1. For example, the reciprocal of $\frac{3}{4}$ is $\frac{4}{3}$ and the reciprocal of $\frac{2}{9}$ is $\frac{9}{2}$. Also, the reciprocal of a negative number greater than or equal to -1 is a negative number less than or equal to -1. For example, the reciprocal of $-\frac{7}{8}$ is $-\frac{8}{7}$ and the reciprocal of $-\frac{1}{5}$ is -5. Therefore:

$$-1 \leq \cos\theta < 0 \;\rightarrow\; \sec\theta \leq -1$$
$$0 < \cos\theta \leq 1 \;\rightarrow\; \sec\theta \geq 1$$

► **The set of secant function values is the set of real numbers that are less than or equal to -1 or greater than or equal to 1, that is, $\{x : x \geq 1 \text{ or } x \leq -1\}$.**

The Cosecant Function

DEFINITION

The **cosecant function** is the set of ordered pairs $(\theta, \csc\theta)$ such that for all θ for which $\sin\theta \neq 0$, $\csc\theta = \frac{1}{\sin\theta}$.

We know that $\sin\theta = 0$ for $\theta = 0$, for $\theta = 180$, and for any value of θ that differs from 0 or 180 by 360. Therefore, $\sin\theta = 0$ for all values of θ such that $\theta = 180n$ for all integral values of n. Therefore, $\csc\theta$ is defined for the degree measures of angles, θ, such that $\theta \neq 180n$.

The cosecant function values are the reciprocals of the sine function values. We know that $-1 \leq \sin\theta \leq 1$. Therefore, the cosecant function values is the same set of values as the secant function values. Therefore:

$$-1 \leq \sin\theta < 0 \;\rightarrow\; \csc\theta \leq -1$$

$$0 < \sin\theta \leq 1 \;\rightarrow\; \csc\theta \geq 1$$

► **The set of cosecant function values is the set of real number that are less than or equal to −1 or greater than or equal to 1, that is, $\{x : x \geq 1 \text{ or } x \leq -1\}$.**

The Cotangent Function

DEFINITION

The **cotangent function** is the set of ordered pairs $(\theta, \cot\theta)$ such that for all θ for which $\tan\theta$ is defined and not equal to 0, $\cot\theta = \frac{1}{\tan\theta}$ and for all θ for which $\tan\theta$ is undefined, $\cot\theta = 0$.

We know that $\tan\theta = 0$ for $\theta = 0$, for $\theta = 180$, and for any value of θ that differs from 0 or 180 by 360. Therefore, $\tan\theta = 0$ for all values of θ such that $\theta = 180n$, when n is an integer. Therefore, $\cot\theta$ is defined for the degree measures of angles, θ, such that $\theta \neq 180n$.

The cotangent function values are the real numbers that are 0 and the reciprocals of the nonzero tangent function values. We know that the set of tangent function values is the set of all real numbers. The reciprocal of any nonzero real number is a nonzero real number. Therefore, the set of cotangent function values is the set of real numbers. Therefore:

$$\tan\theta < 0 \;\rightarrow\; \cot\theta < 0$$

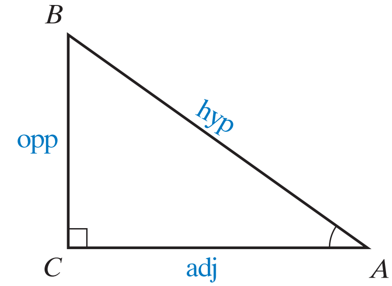
$$\tan\theta > 0 \;\rightarrow\; \cot\theta > 0$$

$$\tan\theta \text{ is undefined} \;\rightarrow\; \cot\theta = 0$$

► **The set of cotangent function values is the set of real numbers.**

Function Values in the Right Triangle

In right triangle ABC, $\angle C$ is a right angle, $\overline{AB}$ is the hypotenuse, $\overline{BC}$ is the side opposite $\angle A$, and $\overline{AC}$ is the side adjacent to $\angle A$. We can express the three reciprocal function values in terms of these sides.



$$\sin A = \frac{\text{opp}}{\text{hyp}} = \frac{BC}{AB} \qquad \csc A = \frac{1}{\sin A} = \frac{1}{\frac{\text{opp}}{\text{hyp}}} = \frac{\text{hyp}}{\text{opp}} = \frac{AB}{BC}$$

$$\cos A = \frac{\text{adj}}{\text{hyp}} = \frac{AC}{AB} \qquad \sec A = \frac{1}{\cos A} = \frac{1}{\frac{\text{adj}}{\text{hyp}}} = \frac{\text{hyp}}{\text{adj}} = \frac{AB}{AC}$$

$$\tan A = \frac{\text{opp}}{\text{adj}} = \frac{BC}{AC} \qquad \cot A = \frac{1}{\tan A} = \frac{1}{\frac{\text{opp}}{\text{adj}}} = \frac{\text{adj}}{\text{opp}} = \frac{AC}{BC}$$

EXAMPLE 1

The terminal side of an angle in standard position intersects the unit circle at $P\left(\frac{5}{7}, -\frac{2\sqrt{6}}{7}\right)$. If the measure of the angle is θ, find the six trigonometric function values.

Solution If the coordinates of the point at which the terminal side of the angle intersects the unit circle are $\left(\frac{5}{7}, -\frac{2\sqrt{6}}{7}\right)$, then:

$$\sin\theta = -\frac{2\sqrt{6}}{7} \qquad \csc\theta = -\frac{7}{2\sqrt{6}} = -\frac{7}{2\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}} = -\frac{7\sqrt{6}}{12}$$

$$\cos\theta = \frac{5}{7} \qquad \sec\theta = \frac{7}{5}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{2\sqrt{6}}{7}}{\frac{5}{7}} = -\frac{2\sqrt{6}}{5} \qquad \cot\theta = \frac{1}{\tan\theta} = -\frac{5}{2\sqrt{6}} = -\frac{5}{2\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}} = \frac{5\sqrt{6}}{12}$$

EXAMPLE 2

Show that $(\tan\theta)(\csc\theta) = \sec\theta$.

Solution Write $\tan\theta$ and $\csc\theta$ in terms of $\sin\theta$ and $\cos\theta$.

$$(\tan\theta)(\csc\theta) = \frac{\sin\theta}{\cos\theta}\left(\frac{1}{\sin\theta}\right) = \frac{1}{\cos\theta} = \sec\theta$$

Exercises

Writing About Mathematics

1. Explain why sec θ cannot equal 0.5.

2. When tan θ is undefined, cot θ is defined to be equal to 0. Use the fact that $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\cot\theta = \frac{1}{\tan\theta}$ to explain why it is reasonable to define cot 90 = 0.

Developing Skills

In 3–10, the terminal side of $\angle ROP$ in standard position intersects the unit circle at P. If m$\angle ROP$ is θ, find: **a.** sin θ **b.** cos θ **c.** tan θ **d.** sec θ **e.** csc θ **f.** cot θ

3. $P(0.6, 0.8)$ **4.** $P(0.96, -0.28)$ **5.** $P\left(-\frac{1}{6}, \frac{\sqrt{35}}{6}\right)$ **6.** $P\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

7. $P\left(\frac{2\sqrt{2}}{3}, \frac{1}{3}\right)$ **8.** $P\left(-\frac{\sqrt{5}}{3}, -\frac{2}{3}\right)$ **9.** $P\left(-\frac{\sqrt{7}}{5}, \frac{3\sqrt{2}}{5}\right)$ **10.** $P\left(-\frac{2\sqrt{10}}{7}, -\frac{3}{7}\right)$

In 11–18, P is a point on the terminal side of an angle in standard position with measure θ and on a circle with center at the origin and radius r. For each point P, find: **a.** r **b.** csc θ **c.** sec θ **d.** cot θ

11. $(3, 4)$ **12.** $(1, 4)$ **13.** $(-3, -3)$ **14.** $(-5, -5)$

15. $(-6, 6)$ **16.** $(-4, 8)$ **17.** $(9, -9)$ **18.** $(9, -3)$

19. If $\sin\theta = 0$, find all possible values of: **a.** cos θ **b.** tan θ **c.** sec θ

20. If $\cos\theta = 0$, find all possible values of: **a.** sin θ **b.** cot θ **c.** csc θ

21. If tan θ is undefined, find all possible values of: **a.** cos θ **b.** sin θ **c.** cot θ

22. If sec θ is undefined, find all possible values of sin θ.

23. What is the smallest positive value of θ such that $\cos\theta = 0$?

Applying Skills

24. Grace walked within range of a cell phone tower. As soon as her cell phone received a signal, she looked up at the tower. The cotangent of the angle of elevation of the top of the tower is $\frac{1}{10}$. If the top of the tower is 75 feet above the ground, to the nearest foot, how far is she from the cell phone tower?

25. A pole perpendicular to the ground is braced by a wire 13 feet long that is fastened to the ground 5 feet from the base of the pole. The measure of the angle the wire makes with the ground is θ. Find the value of:

a. sec θ **b.** csc θ **c.** cot θ

26. An airplane travels at an altitude of 6 miles. At a point on the ground, the measure of the angle of elevation to the airplane is θ. Find the distance to the plane from the point on the ground when:

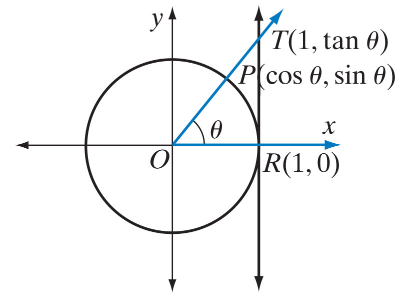
a. $\csc\theta = 2$ **b.** $\sec\theta = \frac{5}{3}$ **c.** $\cot = \sqrt{3}$

27. The equation $\sin^2\theta + \cos^2\theta = 1$ is true for all θ.

a. Use the given equation to prove that $\tan^2\theta + 1 = \sec^2\theta$.

b. Is the equation in part **a** true for all θ? Justify your answer.

c. In the diagram, if $OR = 1$ and $RT = \tan\theta$, the length of what line segment is equal to $\sec\theta$?



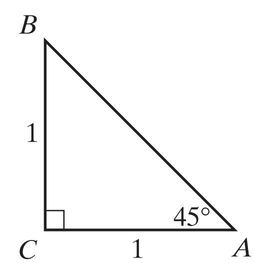
28. Show that $\cot\theta = \frac{\cos\theta}{\sin\theta}$ for all values of θ for which $\sin\theta \neq 0$.

9-6 FUNCTION VALUES OF SPECIAL ANGLES

Equilateral triangles and isosceles right triangles occur frequently in our study of geometry and in the applications of geometry and trigonometry. The angles associated with these triangles are multiples of 30 and 45 degrees. Although a calculator will give rational approximations for the trigonometric function values of these angles, we often need to know their exact values.

The Isosceles Right Triangle

The measure of an acute angle of an isosceles right triangle is 45 degrees. If the measure of a leg is 1, then the length of the hypotenuse can be found by using the Pythagorean Theorem.



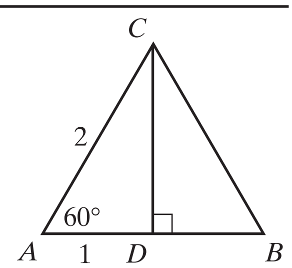
$$c^2 = a^2 + b^2$$
$$AB^2 = 1^2 + 1^2$$
$$AB^2 = 1 + 1$$
$$AB = \sqrt{2}$$

Therefore:

$$\sin 45° = \frac{\text{opp}}{\text{hyp}} = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$
$$\cos 45° = \frac{\text{adj}}{\text{hyp}} = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$
$$\tan 45° = \frac{\text{opp}}{\text{adj}} = \frac{1}{1} = 1$$

The Equilateral Triangle

An equilateral triangle is separated into two congruent right triangles by an altitude from any vertex. In the diagram, $\overline{CD}$ is altitude from C, $\triangle ACD \cong \triangle BCD$, $\angle CDA$ is a right angle, $m\angle A = 60$, and $m\angle ACD = 30$. If $AC = 2$, $AD = 1$.



$$c^2 = a^2 + b^2$$
$$2^2 = 1^2 + CD^2$$
$$4 = 1 + CD^2$$
$$3 = CD^2$$
$$\sqrt{3} = CD$$

Therefore:

$$\sin 30° = \frac{\text{opp}}{\text{hyp}} = \frac{1}{2} \qquad \sin 60° = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{3}}{2}$$

$$\cos 30° = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{3}}{2} \qquad \cos 60° = \frac{\text{adj}}{\text{hyp}} = \frac{1}{2}$$

$$\tan 30° = \frac{\text{opp}}{\text{adj}} = \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3} \qquad \tan 60° = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

The 0°, 30°, 45°, 60°, and 90° angles and their multiples occur frequently and it is useful to memorize the exact trigonometric function values summarized below:

θ	0°	30°	45°	60°	90°
sin θ	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan θ	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

EXAMPLE 1

Use a counterexample to show that $\sin 2\theta \neq 2 \sin \theta$.

Solution Let $\theta = 30°$ and $2\theta = 3(30°) = 60°$.

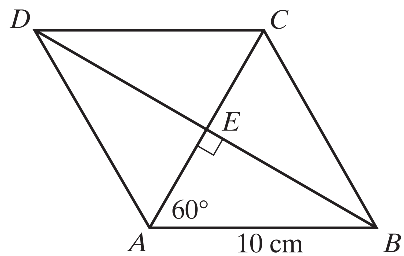
$\sin 2\theta = \sin 60° = \frac{\sqrt{3}}{2}$ and $2 \sin \theta = 2 \sin 30° = 2\left(\frac{1}{2}\right) = 1$.

Therefore, there exists at least one value of θ for which $\sin 2\theta \neq 2 \sin \theta$.

EXAMPLE 2

What are the lengths of the diagonals of a rhombus if the measure of a side is 10 centimeters and the measure of an angle is 120 degrees?

Solution The diagonals of a rhombus are perpendicular, bisect each other, and bisect the angles of the rhombus. Therefore, the diagonals of this rhombus separate it into four congruent right triangles.



Let $ABCD$ be the given rhombus with E the midpoint of $\overline{AC}$ and $\overline{BD}$, $m\angle DAB = 120$ and $AB = 10$ centimeters. Therefore, $\angle AEB$ is a right angle and $m\angle EAB = 60$.

$$\cos 60 = \frac{AE}{AB} \qquad \sin 60 = \frac{BE}{AB}$$
$$\frac{1}{2} = \frac{AE}{10} \qquad \frac{\sqrt{3}}{2} = \frac{BE}{10}$$
$$2AE = 10 \qquad 2BE = 10\sqrt{3}$$
$$AE = 5 \qquad BE = 5\sqrt{3}$$

Answers $AC = 2AE = 2(5) = 10$

$BD = 2BE = 2(5\sqrt{3}) = 10\sqrt{3}$

EXAMPLE 3

What is the exact value of sec 30°?

Solution $\sec 30° = \frac{1}{\cos 30°} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$ *Answer*

Exercises

Writing About Mathematics

1. R is the point $(1, 0)$, P' is the point on a circle with center at the origin, O, and radius r, and $m\angle ROP' = \theta$. Alicia said that the coordinates of P' are $(r \cos \theta, r \sin \theta)$. Do you agree with Alicia? Explain why or why not.

2. Hannah said that if $\cos \theta = a$, then $\sin \theta = \pm\sqrt{1 - a^2}$. Do you agree with Hannah? Explain why or why not.

Developing Skills

In 3–44, find the exact value.

3. cos 30° **4.** sin 30° **5.** csc 30° **6.** tan 30° **7.** cot 30°

8. cos 60° **9.** sec 60° **10.** sin 60° **11.** csc 60° **12.** tan 60°

13. cot 60° **14.** cos 45° **15.** sec 45° **16.** sin 45° **17.** csc 45°

18. tan 45° **19.** cot 45° **20.** cos 180° **21.** sec 180° **22.** sin 180°

23. csc 180° **24.** tan 180° **25.** cot 180° **26.** cos 270° **27.** sec 270°

28. sin 270° **29.** csc 270° **30.** tan 270° **31.** cot 270° **32.** sin 450°

33. $\sin 0° + \cos 0° + \tan 0°$ **34.** $\sin 45° + \cos 60°$ **35.** $\sin 90° + \cos 0° + \tan 45°$

36. $(\cos 60°)^2 + (\sin 60°)^2$ **37.** $(\sec 45°)^2 - (\tan 45°)^2$ **38.** $(\sin 30°)(\cos 60°)$

39. $(\tan 45°)(\cot 45°)$ **40.** $(\sin 45°)(\cos 45°)(\tan 45°)$ **41.** $(\sin 30°)(\sec 60°)$

42. $\frac{\tan 30°}{\cos 60°}$ **43.** $\frac{\sin 45°}{\cos 45°}$ **44.** $\frac{\sin 30°}{\csc 30°}$

Applying Skills

45. A diagonal path across a rectangular field makes an angle of 30 degrees with the longer side of the field. If the length of the path is 240 feet, find the exact dimensions of the field.

46. Use a counterexample to show that $\sin A + \sin B = \sin (A + B)$ is false.

47. Use a counterexample to show that $A < B$ implies $\cos A < \cos B$ is false.

9-7 FUNCTION VALUES FROM THE CALCULATOR

We used our knowledge of an equilateral triangle and of an isosceles right triangle to find the trigonometric function values for some special angles. How can we find the trigonometric function values of any number?

Before calculators and computers were common tools, people who worked with trigonometric functions used tables that supplied the function values. Now these function values are stored in most calculators.

Compare the values given by a calculator to the exact values for angles of 30°, 45° and 60°. On your calculator, press **MODE**. The third line of that menu lists RADIAN and DEGREE. These are the two common angle measures. In a later chapter, we will work with radians. For now, DEGREE should be highlighted on your calculator.

We know that $\cos 30° = \frac{\sqrt{3}}{2}$. When you enter cos 30° into your calculator, the display will show an approximate decimal value.

ENTER: **COS** 30 **)** **ENTER** **2nd** **√** 3 **)** **÷** 2 **ENTER**

DISPLAY:

```
cos(30)
          .8660254038
√(3)/2
          .8660254038
```

These two displays tell us that 0.8660254038 is a rational approximation of the exact irrational value of cos 30°.

A degree, like an hour, is divided into 60 minutes. The measure of an angle written as 37° 45′ is read 37 degrees, 45 minutes, which is equivalent to $37\frac{45}{60}$ degrees. Use the following calculator key sequence to find the tangent of an angle with this measure.

ENTER: **TAN** 37 **+** 45 **÷** 60 **)** **ENTER**

DISPLAY:

```
tan(37+45/60)
          .7742827278
```

EXAMPLE 1

Find sin 70° to four decimal places.

Solution Use a calculator.

ENTER: SIN 70) ENTER

DISPLAY:

```
sin(70)
          .9396926208
```

Round the decimal value. Since the digit following the fourth decimal place is greater than 4, add 1 to the digit in the fourth decimal place.

Answer sin 70° ≈ 0.9397

EXAMPLE 2

Find sec 54° to four decimal places.

Solution $\sec 54° = \frac{1}{\cos 54°}$

ENTER: 1 ÷ COS 54) ENTER

DISPLAY:

```
1/cos(54)
          1.701301617
```

Round the decimal value. Since the digit following the fourth decimal place is less than 5, drop that digit and all digits that follow.

Answer sec 54° ≈ 1.7013

EXAMPLE 3

Find sin 215° to four decimal places.

Solution ENTER: SIN 215) ENTER

DISPLAY:

```
sin(215)
          -.5735764364
```

Recall that when the degree measure of an angle is greater than 180 and less than 270, the angle is a third-quadrant angle. The *y*-coordinate of the intersection of the terminal side with the unit circle is negative. Therefore, the sine of the angle is negative.

Answer -0.5736

Degree Measures of Angles

If $\sin\theta = 0.9205$, then θ is the measure of an angle whose sine is 0.9205. In a circle, the degree measure of a central angle is equal to the degree measure of the intercepted arc. Therefore, we can also say that θ is the measure of an arc whose sine is 0.9205. This can be written in symbols.

$$\sin\theta = 0.9205 \;\rightarrow\; \theta = \text{the angle whose sine is } 0.9205$$
$$\sin\theta = 0.9205 \;\rightarrow\; \theta = \text{the arc whose sine is } 0.9205$$

The words "the arc whose sine is" are abbreviated as **arcsine**. We further shorten arcsine to arcsin.

$$\sin\theta = 0.9205 \;\rightarrow\; \theta = \arcsin 0.9205$$

On a calculator, the symbol for arcsin is "$\sin^{-1}$."

$$\sin\theta = 0.9205 \;\rightarrow\; \theta = \sin^{-1} 0.9205$$

We can also write:

$$\cos\theta = 0.3907 \;\rightarrow\; \theta = \arccos 0.3907 = \cos^{-1} 0.3907$$
$$\tan\theta = 2.3559 \;\rightarrow\; \theta = \arctan 2.3559 = \tan^{-1} 2.3559$$

Arccos and $\cos^{-1}$ are abbreviations for **arccosine**. Similarly, arctan and $\tan^{-1}$ are abbreviations for **arctangent**.

EXAMPLE 4

Find θ to the nearest degree if $\cos\theta = 0.8988$ and $0° < \theta < 180°$.

Solution $\cos\theta = 0.8988 \rightarrow \theta = \arccos 0.8988 = \cos^{-1} 0.8988$

ENTER: 2nd $\cos^{-1}$ 0.8988) ENTER

DISPLAY:

```
cos⁻¹(0.8988)
      25.99922183
```

To the nearest degree, $\theta = 26°$. *Answer*

EXAMPLE 5

Find θ to the nearest minute if $\tan\theta = 3.5782$ and $-90° < \theta < 90°$.

Solution The calculator will return the value of θ as a whole number of degrees followed by a decimal value. To change the decimal part of the number to minutes, multiply by 60.

ENTER: 2nd TAN⁻¹ 3.5782) ENTER

DISPLAY:

```
TAN-1(3.5782)
         74.38590988
```

Now subtract 74 from this answer and multiply the difference by 60.

ENTER: (2nd ANS − 74) × 60 ENTER

DISPLAY:

```
TAN-1(3.5782)
         74.38590988
(Ans-74)*60
         23.15459294
```

To the nearest minute, $\theta = 74°\ 23'$. *Answer*

Exercises

Writing About Mathematics

1. Explain why the calculator displays an error message when TAN 90 is entered.

2. Explain why the calculator displays the same value for sin 400° as for sin 40°.

Developing Skills

In 3–38, find each function value to four decimal places.

3. sin 28°	**4.** cos 35°	**5.** tan 78°	**6.** cos 100°
7. sin 170°	**8.** tan 200°	**9.** tan 20°	**10.** cos 255°
11. cos 75°	**12.** sin 280°	**13.** sin 80°	**14.** tan 375°
15. tan 15°	**16.** cos 485°	**17.** cos 125°	**18.** sin (−10°)
19. sin 350°	**20.** sin 190°	**21.** cos 18° 12′	**22.** sin 57° 40′

23. $\tan 88° \, 30'$ **24.** $\sin 105° \, 50'$ **25.** $\tan 172° \, 18'$ **26.** $\cos 205° \, 12'$

27. $\sin 205° \, 12'$ **28.** $\tan 266° \, 27'$ **29.** $\sec 72°$ **30.** $\csc 15°$

31. $\cot 63°$ **32.** $\sec 100°$ **33.** $\csc 125°$ **34.** $\cot 165°$

35. $\csc 245°$ **36.** $\cot 254°$ **37.** $\sec 307°$ **38.** $\csc 347°$

In 39–50, find the smallest positive value of θ to the nearest degree.

39. $\sin \theta = 0.3455$ **40.** $\cos \theta = 0.4383$ **41.** $\tan \theta = 0.2126$ **42.** $\cos \theta = 0.7660$

43. $\tan \theta = 0.7000$ **44.** $\sin \theta = 0.9990$ **45.** $\cos \theta = 0.9990$ **46.** $\tan \theta = 1.8808$

47. $\sin \theta = 0.5446$ **48.** $\cos \theta = 0.5446$ **49.** $\tan \theta = 1.0355$ **50.** $\tan \theta = 12.0000$

In 51–58, find the smallest positive value of θ to the nearest minute.

51. $\sin \theta = 0.2672$ **52.** $\cos \theta = 0.2672$ **53.** $\sin \theta = 0.9692$ **54.** $\cos \theta = 0.9692$

55. $\sin \theta = 0.6534$ **56.** $\cos \theta = 0.6534$ **57.** $\tan \theta = 7.3478$ **58.** $\tan \theta = 0.0892$

Applying Skills

59. A ramp that is 12 feet long is used to reach a doorway that is 3.5 feet above the level ground. Find, to the nearest degree, the measure the ramp makes with the ground.

60. The bed of a truck is 4.2 feet above the ground. In order to unload boxes from the truck, the driver places a board that is 12 feet long from the bed of the truck to the ground. Find, to the nearest minute, the measure the board makes with the ground.

61. Three roads intersect to enclose a small triangular park. A path that is 72 feet long extends from the intersection of two of the roads to the third road. The path is perpendicular to that road at a point 65 feet from one of the other intersections and 58 feet from the third. Find, to the nearest ten minutes, the measures of the angles at which the roads intersect.

62. The terminal side of an angle in standard position intersects the unit circle at the point $(-0.8, 0.6)$.

a. In what quadrant does the terminal side of the angle lie?

b. Find, to the nearest degree, the smallest positive measure of the angle.

63. The terminal side of an angle in standard position intersects the unit circle at the point $(0.28, -0.96)$.

a. In what quadrant does the terminal side of the angle lie?

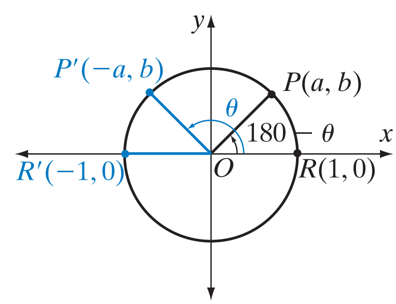
b. Find, to the nearest degree, the smallest positive measure of the angle.

9-8 REFERENCE ANGLES AND THE CALCULATOR

In Section 9-7, we used the calculator to find the function values of angles. We know that two or more angles can have the same function value. For example, when the degree measures of two angles differ by 360, these angles, in standard position, have the same terminal side and therefore the same function values. In addition to angles whose measures differ by a multiple of 360, are there other angles that have the same function values?

Second-Quadrant Angles

In the diagram, $R(1, 0)$ and $P(a, b)$ are points on the unit circle. Under a reflection in the y-axis, the image of $P(a, b)$ is $P'(-a, b)$ and the image of $R(1, 0)$ is $R'(-1, 0)$. Since angle measure is preserved under a line reflection, $m\angle ROP = m\angle R'OP'$.



The rays $\overrightarrow{OR}$ and $\overrightarrow{OR'}$ are opposite rays and $\angle ROP'$ and $\angle R'OP'$ are supplementary. Therefore, $\angle ROP'$ and $\angle ROP$ are supplementary.

$$m\angle ROP' + m\angle ROP = 180$$

$$m\angle ROP = 180 - m\angle ROP'$$

- If $m\angle ROP' = \theta$, then $m\angle ROP = 180 - \theta$.
- If $m\angle ROP' = \theta$, then $\sin\theta = b$ and $\cos\theta = -a$.
- If $m\angle ROP = (180 - \theta)$, then $\sin(180 - \theta) = b$ and $\cos(180 - \theta) = a$.

Therefore:

$$\sin\theta = \sin(180 - \theta) \qquad \cos\theta = -\cos(180 - \theta)$$

Since $\tan\theta = \frac{\sin\theta}{\cos\theta}$, then also:

$$\tan\theta = \frac{\sin(180 - \theta)}{-\cos(180 - \theta)} = -\tan(180 - \theta)$$

Let θ be the measure of a second-quadrant angle. Then there exists a first-quadrant angle with measure $180 - \theta$ such that:

$$\mathbf{\sin\theta = \sin(180 - \theta)}$$

$$\mathbf{\cos\theta = -\cos(180 - \theta)}$$

$$\mathbf{\tan\theta = -\tan(180 - \theta)}$$

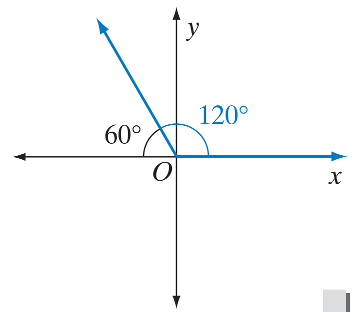
The positive acute angle $(180 - \theta)$ is the **reference angle of the second-quadrant angle**. When drawn in standard position, the reference angle is in the first quadrant.

EXAMPLE 1

Find the exact values of sin 120°, cos 120°, and tan 120°.

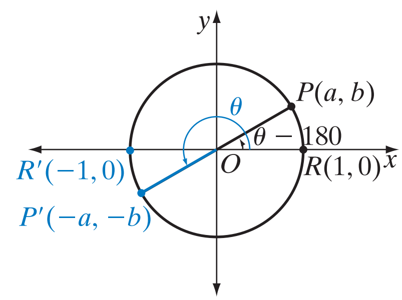
Solution The measure of the reference angle for an angle of 120° is 180° − 120° = 60°.

$$\sin 120° = \sin 60° = \frac{\sqrt{3}}{2}$$
$$\cos 120° = -\cos 60° = -\tfrac{1}{2}$$
$$\tan 120° = -\tan 60° = -\sqrt{3}$$



Third-Quadrant Angles

In the diagram, $R(1, 0)$ and $P(a, b)$ are two points on the unit circle. Under a reflection in the origin, the image of $R(1,0)$ is $R'(-1,0)$ and the image of $P(a, b)$ is $P'(-a, -b)$. Since angle measure is preserved under a point reflection, $\text{m}\angle ROP = \text{m}\angle R'OP'$.



$$\begin{aligned}\text{m}\angle ROP' &= 180 + \text{m}\angle R'OP' \\ &= 180 + \text{m}\angle ROP\end{aligned}$$

$$\text{m}\angle ROP = \text{m}\angle ROP' - 180$$

- If $\text{m}\angle ROP' = \theta$, then $\text{m}\angle ROP = \theta - 180$.
- If $\text{m}\angle ROP' = \theta$, then $\sin\theta = -b$ and $\cos\theta = -a$.
- If $\text{m}\angle ROP = (\theta - 180)$, then $\sin(\theta - 180) = b$ and $\cos(\theta - 180) = a$.

Therefore:

$$\sin\theta = -\sin(\theta - 180) \qquad \cos\theta = -\cos(\theta - 180)$$

Since $\tan\theta = \frac{\sin\theta}{\cos\theta}$, then:

$$\tan\theta = \frac{-\sin(\theta - 180)}{-\cos(\theta - 180)} = \tan(\theta - 180)$$

Let θ be the measure of a third-quadrant angle. Then there exists a first-quadrant angle with measure $\theta - 180$ such that:

$$\boldsymbol{\sin\theta = -\sin(\theta - 180)}$$
$$\boldsymbol{\cos\theta = -\cos(\theta - 180)}$$
$$\boldsymbol{\tan\theta = \tan(\theta - 180)}$$

The positive acute angle $(\theta - 180)$ is the **reference angle of the third-quadrant angle**. When drawn in standard position, the reference angle is in the first quadrant.

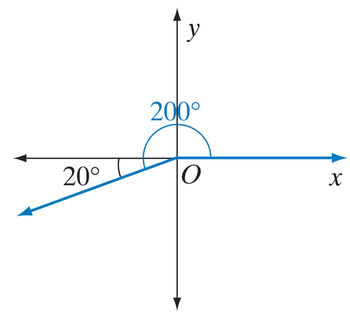
EXAMPLE 2

Express sin 200°, cos 200°, and tan 200° in terms of a function value of an acute angle.

Solution An angle of 200° is in the third quadrant.

For a third-quadrant angle with measure θ, the measure of the reference angle is $\theta - 180$.

The measure of the reference angle for an angle of 200° is $200° - 180° = 20°$.



$$\sin 200° = -\sin 20°$$
$$\cos 200° = -\cos 20°$$
$$\tan 200° = \tan 20°$$

ENTER: SIN 200) ENTER

COS 200) ENTER

TAN 200) ENTER

DISPLAY:

```
sin(200)
        -.3420201433
cos(200)
        -.9396926208
tan(200)
         .3639702343
```

ENTER: (-) SIN 20) ENTER

(-) COS 20) ENTER

TAN 20) ENTER

DISPLAY:

```
-sin(20)
        -.3420201433
-cos(20)
        -.9396926208
tan(20)
         .3639702343
```

Answer $\sin 200° = -\sin 20°$, $\cos 200° = -\cos 20°$, $\tan 200° = \tan 20°$

EXAMPLE 3

If $180 < \theta < 270$ and $\sin \theta = -0.5726$, what is the value of θ to the nearest degree?

Solution If $m\angle A = \theta$, $\angle A$ is a third-quadrant angle. The measure of the reference angle is $\theta - 180$ and $\sin (\theta - 180) = 0.5726$.

ENTER: **2nd** **SIN^{-1}** 0.5726 **)** **ENTER**

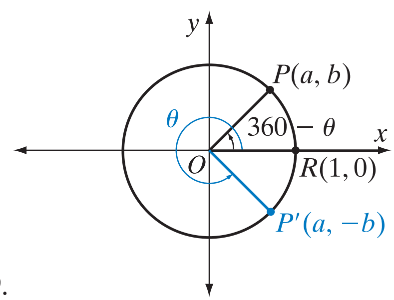
DISPLAY:
```
sin⁻¹(0.5726)
          34.9317314
```

To the nearest degree, $(\theta - 180) = 35$. Therefore:

$$\theta = 35 + 180 = 215° \quad \textit{Answer}$$

Fourth-Quadrant Angles

In the diagram, $R(1, 0)$ and $P(a, b)$ are points on the unit circle. Under a reflection in the x-axis, the image of $P(a, b)$ is $P'(a, -b)$ and the image of $R(1, 0)$ is $R(1, 0)$. Since angle measure is preserved under a line reflection, the measure of the acute angle $\angle ROP$ is equal to the measure of the acute angle $\angle ROP'$.



- If $m\angle ROP' = \theta$, then $m\angle ROP = 360 - \theta$.
- If $m\angle ROP' = \theta$, then $\sin \theta = -b$ and $\cos \theta = a$.
- If $m\angle ROP = (360 - \theta)$, then $\sin (360 - \theta) = b$ and $\cos (360 - \theta) = a$.

Therefore:

$$\sin \theta = -\sin (360 - \theta) \qquad \cos \theta = \cos (360 - \theta)$$

Since $\tan \theta = \frac{\sin \theta}{\cos \theta}$, then:

$$\tan \theta = \frac{-\sin (360 - \theta)}{\cos (360 - \theta)} = -\tan (360 - \theta)$$

Let θ be the measure of a fourth-quadrant angle. Then there exists a first-quadrant angle with measure $360 - \theta$ such that:

$$\boldsymbol{\sin \theta = -\sin (360 - \theta)}$$

$$\boldsymbol{\cos \theta = \cos (360 - \theta)}$$

$$\boldsymbol{\tan \theta = -\tan (360 - \theta)}$$

The positive acute angle $(360 - \theta)$ is the **reference angle of the fourth-quadrant angle**. When drawn in standard position, the reference angle is in the first quadrant.

EXAMPLE 4

If $\sin\theta = -0.7424$, find, to the nearest degree, two positive values of θ that are less than 360°.

Solution Find the reference angle by finding arcsin 0.7424.

ENTER: 2nd SIN⁻¹ .7424) ENTER

DISPLAY:

```
sin⁻¹(.7424)
         47.93626204
```

To the nearest degree, the measure of the reference angle is 48°. The sine is negative in the third quadrant and in the fourth quadrant.

In the third quadrant:	In the fourth quadrant:
$48 = \theta - 180$	$48 = 360 - \theta$
$\theta = 228°$	$\theta = 360 - 48$
	$\theta = 312°$

Alternative Solution θ = the angle whose sine is -0.7424 and arcsin $-0.7424 = \sin^{-1} -0.7424$.

ENTER: 2nd SIN⁻¹ (-) .7424) ENTER

DISPLAY:

```
sin⁻¹(⁻.7424)
        ⁻47.93626204
```

There are many angles whose sine is -0.7424. The calculator returns the measure of the angle with the smallest absolute value. When the sine of the angle is negative, this is an angle with a negative degree measure, that is, an angle formed by a clockwise rotation. To find the positive measure of an angle with the same terminal side, add 360: $\theta = -48 + 360 = 312°$. This is a fourth-quadrant angle whose reference angle is $360 - 312 = 48$. There is also a third-quadrant angle whose sine is -0.7424 and whose reference angle is 48. In the third quadrant:

$$48 = \theta + 180$$

$$\theta = 228°$$

Answer 228° and 312°

SUMMARY

Let θ be the measure of an angle $90° < \theta < 360°$.

	Second Quadrant	Third Quadrant	Fourth Quadrant
Reference Angle	$180 - \theta$	$\theta - 180$	$360 - \theta$
sin θ	$\sin(180 - \theta)$	$-\sin(\theta - 180)$	$-\sin(360 - \theta)$
cos θ	$-\cos(180 - \theta)$	$-\cos(\theta - 180)$	$\cos(360 - \theta)$
tan θ	$-\tan(180 - \theta)$	$\tan(\theta - 180)$	$-\tan(360 - \theta)$

Exercises

Writing About Mathematics

1. Liam said that if $0 < \theta < 90$, then when the degree measure of a fourth-quadrant angle is $-\theta$, the degree measure of the reference angle is θ. Do you agree with Liam? Explain why or why not.

2. Sammy said that if a negative value is entered for $\sin^{-1}$, $\cos^{-1}$, or $\tan^{-1}$, the calculator will return a negative value for the measure of the angle. Do you agree with Sammy? Explain why or why not.

Developing Skills

In 3–7, for each angle with the given degree measure: **a.** Draw the angle in standard position. **b.** Draw its reference angle as an acute angle formed by the terminal side of the angle and the x-axis. **c.** Draw the reference angle in standard position. **d.** Give the measure of the reference angle.

3. 120° **4.** 250° **5.** 320° **6.** −45° **7.** 405°

In 8–17, for each angle with the given degree measure, find the measure of the reference angle.

8. 100° **9.** 175° **10.** 210° **11.** 250° **12.** 285°
13. 310° **14.** 95° **15.** 290° **16.** −130° **17.** 505°

In 18–27, express each given function value in terms of a function value of a positive acute angle (the reference angle).

18. sin 215° **19.** cos 95° **20.** tan 255° **21.** cos 312° **22.** tan 170°
23. sin 285° **24.** cos 245° **25.** tan 305° **26.** sin −56° **27.** sin 500°

In 28–43, for each function value, if $0° \leq \theta < 360°$, find, to the nearest degree, two values of θ.

28. $\sin\theta = 0.3420$ **29.** $\cos\theta = 0.6283$ **30.** $\tan\theta = 0.3240$ **31.** $\tan\theta = 1.4281$
32. $\sin\theta = 0.8090$ **33.** $\sin\theta = -0.0523$ **34.** $\cos\theta = -0.3090$ **35.** $\cos\theta = 0.9205$
36. $\tan\theta = -9.5141$ **37.** $\sin\theta = 0.2419$ **38.** $\cos\theta = -0.7431$ **39.** $\sin\theta = -0.1392$
40. $\tan\theta = -0.1405$ **41.** $\sin\theta = 0$ **42.** $\cos\theta = 0$ **43.** $\tan\theta = 0$

CHAPTER SUMMARY

In $\triangle ABC$ with a right angle at C, $\overline{BC}$ is the leg that is opposite $\angle A$, $\overline{AC}$ is the leg that is adjacent to $\angle A$, and $\overline{AB}$ is the hypotenuse.

$$\sin A = \frac{\text{opp}}{\text{hyp}} \qquad \cos A = \frac{\text{adj}}{\text{hyp}} \qquad \tan A = \frac{\text{opp}}{\text{adj}}$$

An angle is in **standard position** when its vertex is at the origin and its **initial side** is the nonnegative ray of the x-axis. The measure of an angle in standard position is positive when the rotation from the initial side to the **terminal side** is in the counterclockwise direction. The measure of an angle in standard position is negative when the rotation from the initial side to the terminal side is in the clockwise direction.

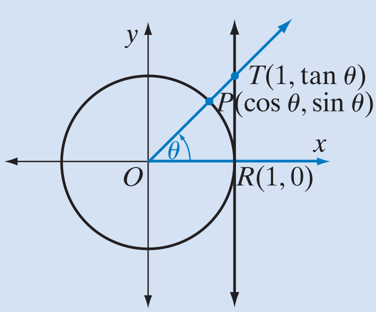
We classify angles in standard position according to the quadrant in which the terminal side lies.

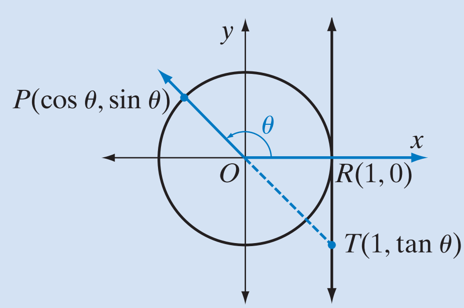
- If $0 < \theta < 90$ and $m\angle AOB = \theta + 360n$ for any integer n, then $\angle AOB$ is a first-quadrant angle.
- If $90 < \theta < 180$ and $m\angle AOB = \theta + 360n$ for any integer n, then $\angle AOB$ is a second-quadrant angle.
- If $180 < \theta < 270$ and $m\angle AOB = \theta + 360n$ for any integer n, then $\angle AOB$ is a third-quadrant angle.
- If $270 < \theta < 360$ and $m\angle AOB = \theta + 360n$ for any integer n, then $\angle AOB$ is a fourth-quadrant angle.

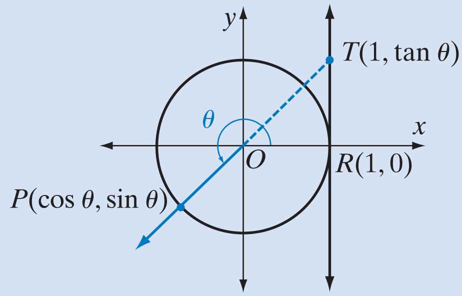
An angle in standard position whose terminal side lies on either the x-axis or the y-axis is called a **quadrantal angle**. The measure of a quadrantal angle is a multiple of 90.

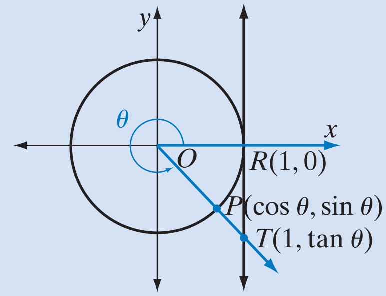
Angles in standard position that have the same terminal side are **coterminal angles**.

A circle with center at the origin and radius 1 is the **unit circle** and has the equation $x^2 + y^2 = 1$.









Let $P(p, q)$ be the point at which the terminal side of an angle in standard position intersects the unit circle, and $T(1, t)$ be the point at which the line tangent to the circle at $R(1, 0)$ intersects the terminal side of the angle. Let $m\angle ROP = \theta$.

- The **sine function** is a set of ordered pairs $(\theta, \sin\theta)$ such that $\sin\theta = q$.
- The **cosine function** is a set of ordered pairs $(\theta, \cos\theta)$ such that $\cos\theta = p$.
- The **tangent function** is the set of ordered pairs $(\theta, \tan\theta)$ such that $\tan\theta = t$.
- The **secant function** is the set of ordered pairs $(\theta, \sec\theta)$ such that for all θ for which $\cos\theta \neq 0$, $\sec\theta = \frac{1}{\cos\theta}$.
- The **cosecant function** is the set of ordered pairs $(\theta, \csc\theta)$ such that for all θ for which $\sin\theta \neq 0$, $\csc\theta = \frac{1}{\sin\theta}$.
- The **cotangent function** is the set of ordered pairs $(\theta, \cot\theta)$ such that for all θ for which $\tan\theta$ is defined and not equal to 0, $\cot\theta = \frac{1}{\tan\theta}$ and for all θ for which $\tan\theta$ is undefined, $\cot\theta = 0$.

The secant, cosecant, and cotangent functions are called **reciprocal functions**.

The equilateral triangle and the isosceles right triangle make it possible to find exact values for angles of 30°, 45°, and 60°.

θ	0°	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

The rational approximation of the values of the sine, cosine, and tangent functions can be displayed by most calculators.

If $\sin\theta = a$, then $\theta =$ **arcsin *a*** or $\theta = \sin^{-1} a$, read as "θ is the angle whose measure is a." We can also write $\tan\theta = a$ as $\theta =$ **arctan *a*** $= \tan^{-1} a$ and $\cos\theta = a$ as $\theta =$ **arccos *a*** $= \cos^{-1} a$.

A minute is $\frac{1}{60}$ of a degree. To express the measure of an angle in degrees and minutes, multiply the decimal part of the degree measure by 60.

The trigonometric function values of angles with degree measure greater than 90 or less than 0 can be found from their values at corresponding acute angles called **reference angles**.

	Second Quadrant	Third Quadrant	Fourth Quadrant
Reference Angle	$180 - \theta$	$\theta - 180$	$360 - \theta$
sin θ	$\sin(180 - \theta)$	$-\sin(\theta - 180)$	$-\sin(360 - \theta)$
cos θ	$-\cos(180 - \theta)$	$-\cos(\theta - 180)$	$\cos(360 - \theta)$
tan θ	$-\tan(180 - \theta)$	$\tan(\theta - 180)$	$-\tan(360 - \theta)$

VOCABULARY

9-1 Hypotenuse • Leg • Similar triangles • Sine • Cosine • Tangent

9-2 Initial side • Terminal side • Standard position • θ (theta) • Quadrantal angle • Coterminal angle • Angular speed

9-3 Unit circle • Sine function • Cosine function

9-4 Tangent function

9-5 Reciprocal function • Secant function • Cosecant function • Cotangent function

9-7 Arcsine • Arccosine • Arctangent

9-8 Reference angle

REVIEW EXERCISES

In 1–5: **a.** Draw each angle in standard position. **b.** Draw its reference angle as an acute angle formed by the terminal side of the angle and the x-axis. **c.** Draw the reference angle in standard position. **d.** Give the measure of the reference angle.

1. $220°$ **2.** $300°$ **3.** $145°$ **4.** $-100°$ **5.** $600°$

6. For each given angle, find a coterminal angle with a measure of θ such that $0° \leq \theta < 360°$: **a.** $505°$ **b.** $-302°$

In 7–10, the terminal side of $\angle ROP$ in standard position intersects the unit circle at P. If $m\angle ROP$ is θ, find:

a. the quadrant of $\angle ROP$ **b.** $\sin\theta$ **c.** $\cos\theta$ **d.** $\tan\theta$ **e.** $\csc\theta$ **f.** $\sec\theta$ **g.** $\cot\theta$

7. $P(0.8, -0.6)$ **8.** $P\left(\frac{3}{4}, \frac{\sqrt{7}}{4}\right)$ **9.** $P\left(-\frac{2\sqrt{6}}{5}, \frac{1}{5}\right)$ **10.** $P\left(-\frac{2}{3}, -\frac{\sqrt{5}}{3}\right)$

In 11–14, the given point is on the terminal side of angle θ in standard position. Find: **a.** the quadrant of the angle **b.** $\sin \theta$ **c.** $\cos \theta$ **d.** $\tan \theta$ **e.** $\csc \theta$ **f.** $\sec \theta$ **g.** $\cot \theta$

11. $(12, 9)$ **12.** $(-100, 0)$ **13.** $(-8, -6)$ **14.** $(9, -13)$

15. For each given angle in standard position, determine, to the nearest tenth, the coordinates of the point where the terminal side intersects the unit circle: **a.** 405° **b.** 79°

In 16–27, find each exact function value.

16. cos 30° **17.** sin 60° **18.** tan 45° **19.** tan 120°

20. cos 135° **21.** sin 240° **22.** cos 330° **23.** sin 480°

24. sec 45° **25.** csc 30° **26.** cot 60° **27.** sec 120°

In 28–35, find each function value to four decimal places.

28. cos 50° **29.** tan 80° **30.** sin 110° **31.** tan 230°

32. cos 187° **33.** sin (−24°) **34.** cos (−230°) **35.** tan 730°

In 36–43, express each given function value in terms of a function value of a positive acute angle (the reference angle).

36. cos 100° **37.** sin 300° **38.** cos 280° **39.** tan 210°

40. sin 150° **41.** cos 255° **42.** cos (−40°) **43.** tan (−310°)

In 44–51, for each function value, if $0 \leq \theta < 360$, find, to the nearest degree, two values of θ.

44. $\sin \theta = 0.3747$ **45.** $\cos \theta = 0.9136$

46. $\tan \theta = 1.376$ **47.** $\sin \theta = -0.7000$

48. $\tan \theta = -0.5775$ **49.** $\cos \theta = -0.8192$

50. $\sec \theta = 1.390$ **51.** $\csc \theta = 3.072$

In 52–55, $0 \leq \theta < 360$.

52. For what two values of θ is $\tan \theta$ undefined?

53. For what two values of θ is $\cot \theta$ undefined?

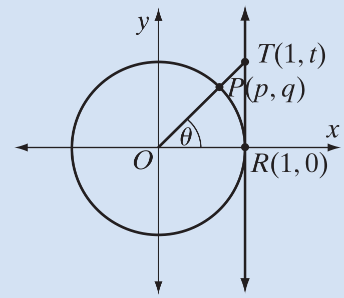
54. For what two values of θ is $\sec \theta$ undefined?

55. For what two values of θ is $\csc \theta$ undefined?

56. In the diagram, m$\angle ROP$ is θ. Express each of the following in terms of a coordinate of R, P, or T.

a. $\sin \theta$ **b.** $\cos \theta$ **c.** $\tan \theta$

d. Use similar triangles to show that $\tan \theta = \frac{\sin \theta}{\cos \theta}$.

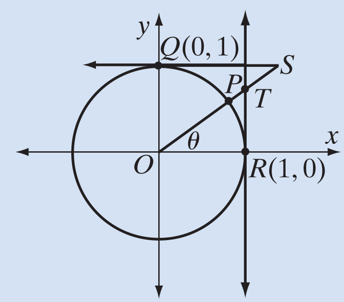


57. A road rises 630 feet every mile. Find to the nearest ten minutes the angle that the road makes with the horizontal.

58. From the top of a building that is 56 feet high, the angle of depression to the base of an adjacent building is 72°. Find, to the nearest foot, the distance between the buildings.

Exploration

In the diagram, $\overrightarrow{OP}$ intersects the unit circle at P. The line that is tangent to the circle at $R(1, 0)$ intersects $\overrightarrow{OP}$ at T. The line that is tangent to the circle at $Q(0,1)$ intersects $\overrightarrow{OP}$ at S. The measure of $\angle ROP$ is θ. Show that $OT = \sec \theta$, $OS = \csc \theta$, and $QS = \cot \theta$. (*Hint:* Use similar triangles.)



CUMULATIVE REVIEW CHAPTERS 1–9

Part I

Answer all questions in this part. Each correct answer will receive 2 credits. No partial credit will be allowed.

1. In simplest form, $(2x + 1)^2 - (x + 2)^2$ is equal to

(1) $3x^2 - 3$
(2) $3x^2 + 8x + 5$
(3) $3x^2 + 5$
(4) $3x^2 + 8x - 3$

2. The solution set of $x^2 - 5x = 6$ is

(1) $\{2, 3\}$
(2) $\{-2, -3\}$
(3) $\{6, -1\}$
(4) $\{-6, 1\}$

3. In simplest form, the fraction $\dfrac{a - \frac{1}{a}}{a + 1}$ is equal to

(1) $-\frac{1}{a}$ (2) $\frac{a - 1}{a}$ (3) $\frac{a + 1}{a}$ (4) $a - 1$

4. The sum of $(3 + \sqrt{12})$ and $(-5 + \sqrt{27})$ is

(1) 16 (2) $-2 + \sqrt{39}$ (3) $-2 + 13\sqrt{3}$ (4) $-2 + 5\sqrt{3}$

5. Which of the following products is a rational number?

(1) $(10 + \sqrt{10})(10 + \sqrt{10})$ (3) $\sqrt{10}(2 + \sqrt{10})$

(2) $(10 + \sqrt{10})(10 - \sqrt{10})$ (4) $10(10 - \sqrt{10})$

6. The fraction $\frac{1 - \sqrt{3}}{1 + \sqrt{3}}$ is equal to

(1) $-\frac{1}{2}$ (2) 2 (3) $2 - \sqrt{3}$ (4) $-2 + \sqrt{3}$

7. Which of the following is a one-to-one function when the domain is the set of real numbers?

(1) $y = x - 5$ (3) $y = x^2 - 2x + 5$

(2) $x^2 + y^2 = 9$ (4) $y = |x - 4|$

8. The sum of the roots of the equation $2x^2 - 5x + 3 = 0$ is

(1) 5 (2) -5 (3) $\frac{5}{2}$ (4) $-\frac{5}{2}$

9. When x and y vary inversely

(1) xy equals a constant. (3) $x + y$ equals a constant.

(2) $\frac{x}{y}$ equals a constant. (4) $x - y$ equals a constant.

10. The expression $2 \log a + \frac{1}{3} \log b$ is equivalent to

(1) $\log \frac{1}{3}a^2b$ (3) $\log (a^2 + \sqrt[3]{b})$

(2) $\log \frac{2}{3}ab$ (4) $\log a^2(\sqrt[3]{b})$

Part II

Answer all questions in this part. Each correct answer will receive 2 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

11. Express the product $(3 - 2i)(-1 + i)$ in $a + bi$ form.

12. Find the solution set of $|2x - 4| < 3$.

Part III

Answer all questions in this part. Each correct answer will receive 4 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

13. Solve for x: $3 + (x + 3)^{\frac{1}{2}} = x$.

14. Write the equation of the circle if the center of the circle is $C(2, 1)$ and one point on the circle is $A(4, 0)$.

Part IV

Answer all questions in this part. Each correct answer will receive 6 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

15. What are the roots of the equation $x^2 + 4 = 6x$?

16. Let $\mathrm{f}(x) = x^2 - x$ and $\mathrm{g}(x) = 5x + 7$.

a. Find $\mathrm{f} \circ \mathrm{g}(-2)$.

b. Write $\mathrm{h}(x) = \mathrm{f} \circ \mathrm{g}(x)$ as a polynomial in simplest form.