4

RELATIONS AND FUNCTIONS

Long-distance truck drivers keep very careful watch on the length of time and the number of miles that they drive each day. They know that this relationship is given by the formula $rt = d$. On some trips, they are required to drive for a fixed length of time each day, and that makes $\frac{d}{r}$ a constant. When this is the case, we say that the distance traveled and the rate of travel are *directly proportional*. When the rate of travel increases, the distance increases and when the rate of travel decreases, the distance decreases. On other trips, truck drivers are required to drive a fixed distance each day, and that makes rt equal a constant. When this the case, we say that the time and the rate of travel are *inversely proportional*. When the rate of travel is increased, the time is decreased and when the rate of travel is decreased, the time is increased. Each of these cases defines a function.

In this chapter, we will study some fundamental properties of functions, and in the chapters that follow, we will apply these properties to some important classes of functions.

4-1 RELATIONS AND FUNCTIONS

Every time that we write an algebraic expression such as $2x + 3$, we are defining a set of ordered pairs, $(x, 2x + 3)$. The expression $2x + 3$ means that for every input x, there is an output that is 3 more than twice x. In **set-builder notation**, we write this set of ordered pairs as $\{(x, y) : y = 2x + 3\}$. This is read "the set of ordered pairs (x, y) such that y is equal to 3 more than twice x." The colon $(:)$ represents the phrase "such that" and the description to the right of the colon defines the elements of the set. Another set of ordered pairs related to the algebraic expression $2x + 3$ is $\{(x, y) : y \geq 2x + 3\}$.
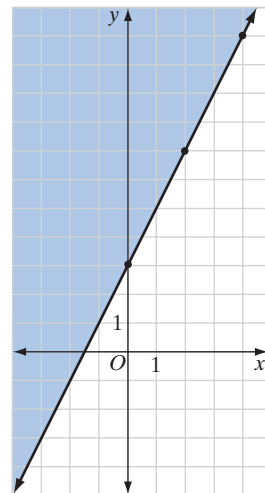
DEFINITION

A **relation** is a set of ordered pairs.

The inequality $y \geq 2x + 3$ defines a relation. For every real number x, there are many possible values of y. The set of ordered pairs that make the inequality true is the union of the pairs that make the equation $y = 2x + 3$ true and the pairs that make the inequality $y > 2x + 3$ true. This is an infinite set. We can list typical pairs from the set but cannot list all pairs.

To show the solution set of the inequality $y \geq 2x + 3$ as points on the coordinate plane, we choose some values of x and y such as $(0, 3)$, $(2, 7)$, and $(4, 11)$ that make the equality true and draw the line through these points. All of the pairs on the line and above the line make the inequality $y \geq 2x + 3$ true.

The equation $y = 2x + 3$ determines a set of ordered pairs. In this relation, for every value of x, there is exactly one value of y. This relation is called a *function*.

DEFINITION

A **function** is a relation in which no two ordered pairs have the same first element.

The ordered pairs of the function $y = 2x + 3$ depend on the set from which the first element, x, can be chosen. The set of values of x is called the **domain of the function**. The set of values for y is called the **range of the function**.

DEFINITION

A **function from set A to set B** is the set of ordered pairs (x, y) such that every $x \in A$ is paired with exactly one $y \in B$.

The set A is the domain of the function. However, the range of the function is *not* necessarily equal to the set B. The range is the set of values of y, a *subset* of B. The symbol $\in$ means "is an element of," so $x \in A$ means "x is an element of set A."

For example, let $\{(x, y) : y = 2x + 3\}$ be a function from the set of integers to the set of integers. Every integer is a value of x but not every integer is a value of y. When x is an integer, $2x + 3$ is an odd integer. Therefore, when the domain of the function $\{(x, y) : y = 2x + 3\}$ is the set of integers, the range is the set of odd integers.

A
(Integers)

B
(Integers)

A
(Real numbers)

B
(Real numbers)
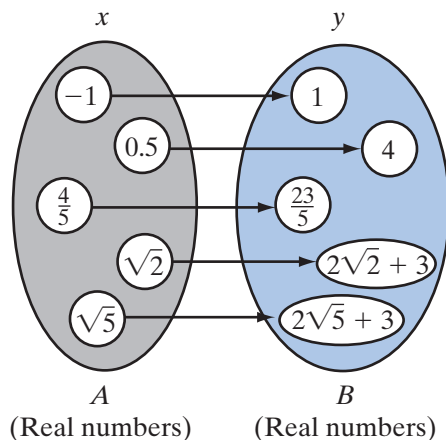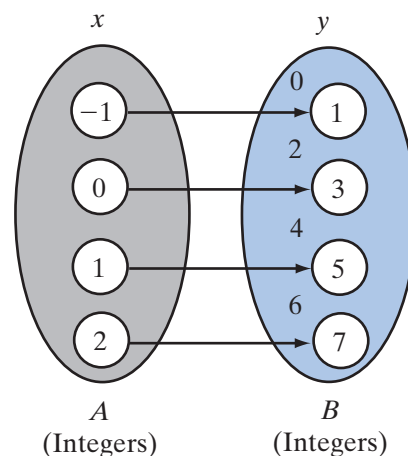
On the other hand, let $\{(x, y) : y = 2x + 3\}$ be a function from set A to set B. Let set A be the set of real numbers and set B be the set of real numbers. In this case, every real number can be written in the form $2x + 3$ for some real number x. Therefore, the range is the set of real numbers. We say that the function $\{(x, y) : y = 2x + 3\}$ is *onto* when the range is equal to set B.

DEFINITION

A function from set A to set B is **onto** if the range of the function is equal to set B.

The following functions have been studied in earlier courses. Unless otherwise stated, each function is from the set of real numbers to the set of real numbers. In general, we will consider set A and set B to be the set of real numbers.

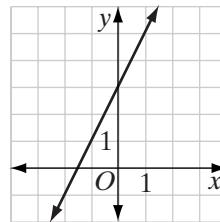| Constant Function |
|---|
| $y = a$, with a a constant. |
| Example: $y = 3$ |
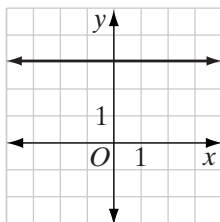
The range is $\{3\}$.

The function is *not* onto.

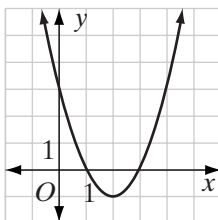| Linear Function |
|---|
| $y = mx + b$, with m and b constants. |
| Example: $y = 2x + 3$ |

The range is the set of real numbers.

The function is onto.

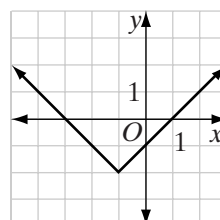| Quadratic Function |
|---|
| $y = ax^2 + bx + c$ with a, b, and c constants and $a \neq 0$. |
| Example: $y = x^2 - 4x + 3$ |

The range is the set of real numbers ≥ -1.

The function is *not* onto.

Absolute Value Function
$y = a + |bx + c|$ with a, b, and c constants.
Example: $y = -2 + |x + 1|$

The range is the set of real numbers ≥ -2.

The function is *not* onto.

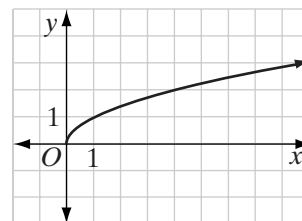| Square Root Function |
|---|
| $$y = \sqrt{ax + b}, \text{ with } ax + b \geq 0.$$ |

- The domain is the set of real numbers greater than or equal to $\frac{-b}{a}$.
- The range is the set of non-negative real numbers.

Example: $y = \sqrt{x}$

This is a function from the set of non-negative real numbers to the set of non-negative real numbers.

The range is the set of non-negative real numbers.

The function is onto.

Vertical Line Test

A relation is a function if no two pairs of the relation have the same first element. On a graph, two points lie on the same vertical line if and only if the coordinates of the points have the same first element. Therefore, if a vertical line intersects the graph of a relation in two or more points, the relation is not a function. This is referred to as the **vertical line test**.

The graph of the relation $\{(x, y) : y = \pm\sqrt{x} \text{ and } x \geq 0\}$ is shown at the right. The graph shows that the line whose equation is $x = 2$ intersects the graph in two points. Every line whose equation is $x = a$ for $a > 0$ also intersects the graph of the relation in two points. Therefore, the relation is not a function.
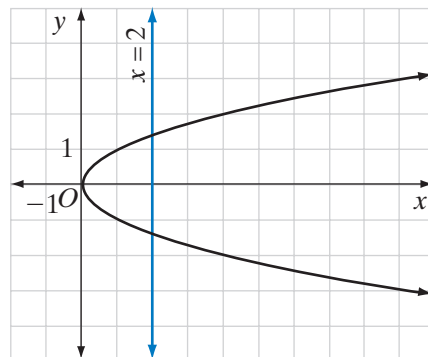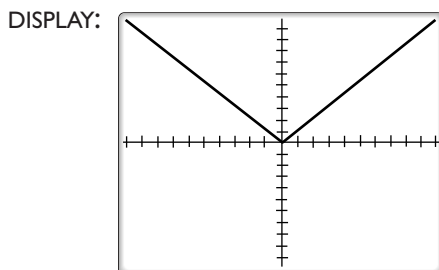

EXAMPLE 1

For each given set, determine:

a. Is the set a relation?　　　　**b.** Is the set a function?

(1) $\{(x, y) : y > x\}$ if $x \in \{\text{real numbers}\}$

(2) $\{x : x > 4\}$ if $x \in \{\text{real numbers}\}$

(3) $\{(x, y) : y = |x|\}$ if $x \in \{\text{real numbers}\}$

Solution (1) **a.** $\{(x, y) : y > x\}$ is a relation because it is a set of ordered pairs.

b. $\{(x, y) : y > x\}$ is not a function because many ordered pairs have the same first elements. For example, $(1, 2)$ and $(1, 3)$ are both ordered pairs of this relation.

(2) **a.** $\{x : x > 4\}$ is not a relation because it is not a set of ordered pairs.

b. $\{x : x > 4\}$ is not a function because it is not a relation.

(3) **a.** $\{(x, y) : y = |x|\}$ is a relation because it is a set of ordered pairs.

b. $\{(x, y) : y = |x|\}$ is a function because for every real number x there is exactly one real number that is the absolute value. Therefore, no two pairs have the same first element.

In part (3) of Example 1, the graph of $y = |x|$ passes the vertical line test, which we can see using a graphing calculator.
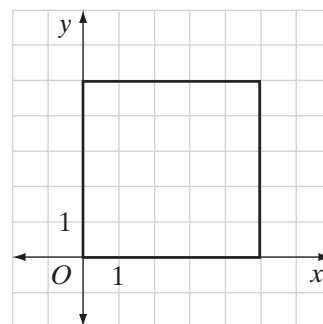
ENTER: `Y=` `MATH` `▶` `1` `X,T,θ,n` `GRAPH`

DISPLAY:

EXAMPLE 2

The graph of a relation is shown at the right.

a. Determine whether or not the graph represents a function.

b. Find the domain of the graph.

c. Find the range of the graph.

Solution **a.** The relation is not a function. Points $(1, 0)$ and $(1, 5)$ as well as many other pairs have the same first element. *Answer*

b. The domain is $\{x : 0 \le x \le 5\}$. *Answer*

c. The range is $\{y : 0 \le y \le 5\}$. *Answer*

EXAMPLE 3

An artist makes hand-painted greeting cards that she sells online for three dollars each. In addition, she charges three dollars for postage for each package of cards that she ships, regardless of size. The total amount, y, that a customer pays for a package of cards is a function of the number of cards purchased, x.

a. Write the relation described above as a function in set-builder notation.

b. What is the domain of the function?

c. Write four ordered pairs of the function.

d. What is the range of the function?

e. Is the function onto?

Solution **a.** The price per card is $3.00, so x cards cost $3x$. The cost of postage, $3.00, is a constant. Therefore, the total cost, y, for x cards is $3x + 3$. In set-builder notation, this is written as:

$$\{(x, y) : y = 3x + 3\}$$

b. The first element, x, which represents the number of cards shipped, must be a positive integer. Therefore, the domain is the set of positive integers.

c. Many ordered pairs are possible. For example:

x	$3x + 3$	y	(x, y)
1	$3(1) + 3$	6	$(1, 6)$
2	$3(2) + 3$	9	$(2, 9)$
3	$3(3) + 3$	12	$(3, 12)$
4	$3(4) + 3$	15	$(4, 15)$

d. Since $3x + 3$ is a multiple of 3, the range is the set of positive integers greater than 3 that are multiples of 3.

e. The domain is the set of positive integers. This is a function from the set of positive integers to the set of positive integers. But the range is the set of positive multiples of 3. The function is not onto because the range is not equal to the set of positive integers.

EXAMPLE 4

Find the largest domain of each function:

a. $y = \dfrac{1}{x + 5}$ **b.** $y = \dfrac{x + 1}{x^2 - 2x - 3}$ **c.** $y = \dfrac{x}{5x^2 + 1}$

Solution The functions are well defined wherever the denominators are *not* equal to 0. Therefore, the domain of each function consists of the set real numbers such that the denominator is not equal to 0.

a. $x + 5 = 0$ or $x = -5$. Therefore, the domain is $\{x : x \neq -5\}$.

b.
$$x^2 - 2x - 3 = 0$$
$$(x - 3)(x + 1) = 0$$
$$x - 3 = 0 \quad | \quad x + 1 = 0$$
$$x = 3 \quad | \quad x = -1$$

Therefore, the domain is $\{x : x \neq -1, 3\}$.

c. The denominator $5x^2 + 1$ cannot be equal to zero. Therefore, the domain is the set of real numbers.

Answers **a.** Domain $= \{x : x \neq -5\}$ **b.** Domain $= \{x : x \neq -1, 3\}$
c. Domain $= \{\text{Real numbers}\}$

Exercises

Writing About Mathematics

**1.** Explain why $\{(x, y) : x = y^2\}$ is not a function but $\{(x, y) : \sqrt{x} = y\}$ is a function.

2. Can $y = \sqrt{x}$ define a function from the set of positive integers to the set of positive integers? Explain why or why not.

Developing Skills

In 3–5: **a.** Explain why each set of ordered pairs is or is not a function. **b.** List the elements of the domain. **c.** List the elements of the range.

3. $\{(1, 1), (2, 4), (3, 9), (4, 16)\}$
4. $\{(1, -1), (0, 0), (1, 1)\}$

5. $\{(-2, 5), (-1, 5), (0, 5), (1, 5), (2, 5)\}$

In 6–11: **a.** Determine whether or not each graph represents a function. **b.** Find the domain for each graph. **c.** Find the range for each graph.

6.

7.

8.

9.

10.

11.
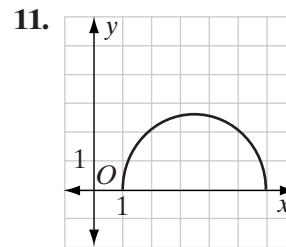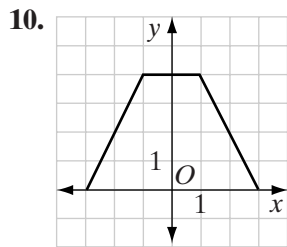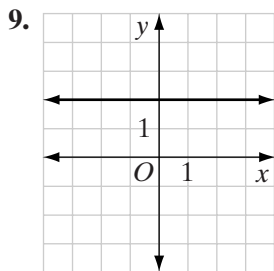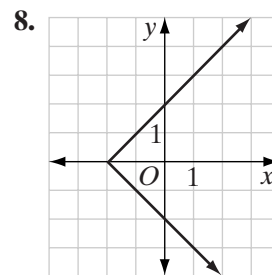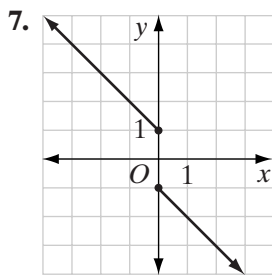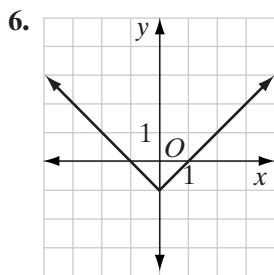
In 12–23, each set is a function from set A to set B. **a.** What is the largest subset of the real numbers that can be set A, the domain of the given function? **b.** If set $A =$ set B, is the function onto? Justify your answer.

12. $\{(x, y) : y = -183\}$
13. $\{(x, y) : y = 5 - x\}$
14. $\{(x, y) : y = x^2\}$

15. $\{(x, y) : y = -x^2 + 3x - 2\}$
16. $\{(x, y) : y = \sqrt{2x}\}$
17. $\{(x, y) : y = |4 - x|\}$

18. $\left\{(x, y) : y = \frac{1}{x}\right\}$
19. $\{(x, y) : y = \sqrt{3 - x}\}$
20. $\left\{(x, y) : y = \frac{1}{\sqrt{x} + 1}\right\}$

21. $\left\{(x, y) : y = \frac{1}{x^2 + 1}\right\}$
22. $\left\{(x, y) : y = \frac{x + 1}{x - 1}\right\}$
23. $\left\{(x, y) : y = \frac{x - 5}{|x - 3|}\right\}$

Applying Skills

24. The perimeter of a rectangle is 12 meters.

  a. If x is the length of the rectangle and y is the area, describe, in set-builder notation, the area as a function of length.

  b. Use integral values of x from 0 to 10 and find the corresponding values of y. Sketch the graph of the function.

  c. What is the largest subset of the real numbers that can be the domain of the function?

25. A candy store sells candy by the piece for 10 cents each. The amount that a customer pays for candy, y, is a function of the number of pieces purchased, x.

  a. Describe, in set-builder notation, the cost in cents of each purchase as a function of the number of pieces of candy purchased.

  b. Yesterday, no customer purchased more than 8 pieces of candy. List the ordered pairs that describe possible purchases yesterday.

  c. What is the domain for yesterday's purchases?

  d. What is the range for yesterday's purchases?

4-2 FUNCTION NOTATION

The rule that defines a function may be expressed in many ways. The letter f, or some other lowercase letter, is often used to designate a function. The symbol $f(x)$ represents the value of the function when evaluated for x. For example, if f is the function that pairs a number with three more than twice that number, f can also be written in any one of the following ways:

1. $f = \{(x, y) : y = 2x + 3\}$ The function f is the set of ordered pairs $(x, 2x + 3)$.

2. $f(x) = 2x + 3$ The function value paired with x is $2x + 3$.

3. $y = 2x + 3$ The second element of the pair (x, y) is $2x + 3$.

4. $\{(x, 2x + 3)\}$ The set of ordered pairs whose second element is 3 more than twice the first.

5. $f: x \rightarrow 2x + 3$ Under the function f, x maps to $2x + 3$.

6. $x \xrightarrow{f} 2x + 3$ The image of x for the function f is $2x + 3$.

The expressions in 2, 3, and 4 are abbreviated forms of that in 1. Note that y and $f(x)$ are both symbols for the second element of an ordered pair of the function whose first element is x. That second element is often called the function value. The equation $f(x) = 2x + 3$ is the most common form of function notation. For the function $f = \{(x, y) : y = 2x + 3\}$, when $x = 5$, $y = 13$ or $f(5) = 13$. The use of parentheses in the symbol f(5) does *not* indicate multipli-

cation. The symbol f(5) represents the function *evaluated* at 5, that is, f(5) is the second element of the ordered pair of the function f when the first element is 5. In the symbol $f(x)$, x can be replaced by any number in the domain of f or by any algebraic expression that represents a number in the domain of f.

Therefore, if $f(x) = 2x + 3$:

1. $f(-4) = 2(-4) + 3 = -5$

2. $f(a + 1) = 2(a + 1) + 3 = 2a + 5$

3. $f(2x) = 2(2x) + 3 = 4x + 3$

4. $f(x^2) = 2x^2 + 3$

We have used f to designate a function throughout this section. Other letters, such as g, h, p, or q can also be used. For example, $g(x) = 2x$ defines a function. Note that if $f(x) = 2x + 3$ and $g(x) = 2x$, then $f(x)$ and $g(x) + 3$ define the same set of ordered pairs and $f(x) = g(x) + 3$.

EXAMPLE I

Let f be the set of ordered pairs such that the second element of each pair is 1 more than twice the first.

a. Write $f(x)$ in terms of x.

b. Write four different ways of symbolizing the function.

c. Find f(7).

Solution **a.** $f(x) = 1 + 2x$

b. $\{(x, y) : y = 1 + 2x\}$ $y = 1 + 2x$

$x \xrightarrow{f} 1 + 2x$ $f: x \rightarrow 1 + 2x$

c. $f(7) = 1 + 2(7) = 15$

Exercises

Writing About Mathematics

1. Let $f(x) = x^2$ and $g(x + 2) = (x + 2)^2$. Are f and g the same function? Explain why or why not.

2. Let $f(x) = x^2$ and $g(x + 2) = x^2 + 2$. Are f and g the same function? Explain why or why not.

Developing Skills

In 3–8, for each function: **a.** Write an expression for f(x). **b.** Find f(5).

3. $y = x - 2$
4. $x \xrightarrow{f} x^2$
5. f: $x \rightarrow |3x - 7|$

6. $\{(x, 5x)\}$
7. $y = \sqrt{x - 1}$
8. $x \xrightarrow{f} \frac{2}{x}$

In 9–14, $y = f(x)$. Find f(−3) for each function.

9. $f(x) = 7 - x$
10. $y = 1 + x^2$
11. $f(x) = -4x$

12. $f(x) = 4$
13. $y = \sqrt{1 - x}$
14. $f(x) = |x + 2|$

15. The graph of the function f is shown at the right.
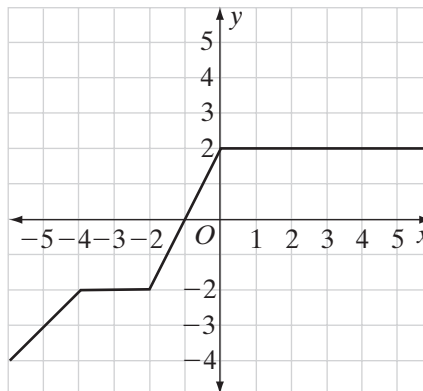
Find:

a. f(2)

b. f(−5)

c. f(−2)

**d.** $f\left(3\frac{1}{2}\right)$

Applying Skills

16. The sales tax t on a purchase is a function of the amount a of the purchase. The sales tax rate in the city of Eastchester is 8%.

a. Write a rule in function notation that can be used to determine the sales tax on a purchase in Eastchester.

b. What is a reasonable domain for this function?

c. Find the sales tax when the purchase is $5.00.

d. Find the sales tax when the purchase is $16.50.

17. A muffin shop's weekly profit is a function of the number of muffins m that it sells. The equation approximating the weekly profit is $f(m) = 0.60m - 900$.

a. Draw the graph showing the relationship between the number of muffins sold and the profit.

b. What is the profit if 2,000 muffins are sold?

c. How many muffins must be sold for the shop to make a profit of $900?

4-3 LINEAR FUNCTIONS AND DIRECT VARIATION

The graph of the **linear function**
$f = \{(x, y) : y = 2x - 1\}$ is shown to the right.

- The graph has a y-intercept of -1, that is, it intersects the y-axis at $(0, -1)$.

- The graph has a slope of 2, that is, the ratio of the change in y to the change in x is $2 : 1$.

- The domain and range of this function are both the set of real numbers.

Since it is a function, no two pairs have the same first element. We know that this is true because no vertical line can intersect the graph of f in more than one point. Note also that no horizontal line can intersect the graph of f in more than one point. This is called the **horizontal line test**. All points with the same second element lie on the same horizontal line, that is, if $f(x_1) = f(x_2)$, then $x_1 = x_2$. Therefore, for the function f, no two pairs have the same second element. We say that the function f is a *one-to-one function*. Every element of the domain is paired with exactly one element of the range and every element of the range is paired with exactly one element of the domain. One-to-one can also be written as 1-1.
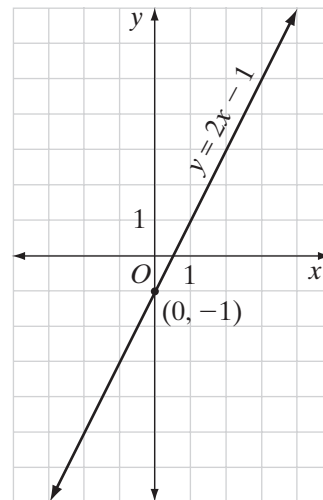
DEFINITION

A function f is said to be **one-to-one** if no two ordered pairs have the same second element.

For any linear function from the set of real numbers to the set of real numbers, the domain and range are the set of real numbers. Since the range is equal to the set of real numbers, a linear function is onto.

▶ **Every non-constant linear function is one-to-one and onto.**

EXAMPLE I

Determine if each of the following functions is one-to-one.

a. $y = 4 - 3x$ **b.** $y = x^2$ **c.** $y = |x + 3|$

*Solution* A function is one-to-one if and only if no two pairs of the function have the same second element.

a. $y = 4 - 3x$ is a linear function whose graph is a slanted line. A slanted line can be intersected by a horizontal line in exactly one point. Therefore, $y = 4 - 3x$ is a one-to-one function.

b. $y = x^2$ is not one-to-one because $(1, 1)$ and $(-1, 1)$ are two ordered pairs of the function.

c. $y = |x + 3|$ is not one-to-one because $(1, 4)$ and $(-7, 4)$ are two ordered pairs of the function.

EXAMPLE 2

Sketch the graph of $y = -2x + 3$ and show that no two points have the same second coordinate.

Solution The graph of $y = -2x + 3$ is a line with a y-intercept of 3 and a slope of -2. Two points have the same second coordinate if and only if they lie on the same horizontal line. A horizontal line intersects a slanted line in exactly one point. In the diagram, the graph of $y = 5$ intersects the graph of $y = -2x + 3$ in exactly one point. The function passes the horizontal line test. No two points on the graph of $y = -2x + 3$ have the same second element and the function is one-to-one.
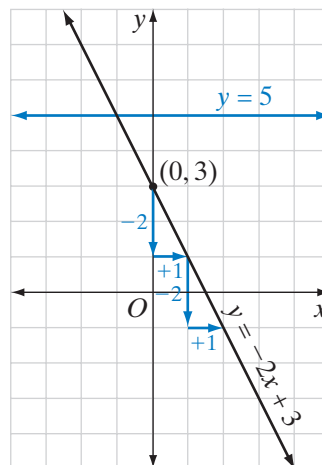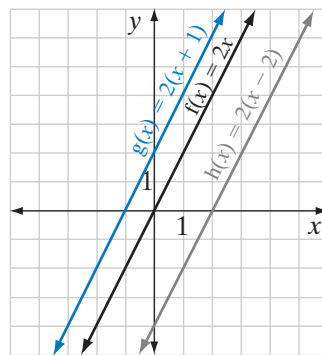

## Transformations of Linear Functions
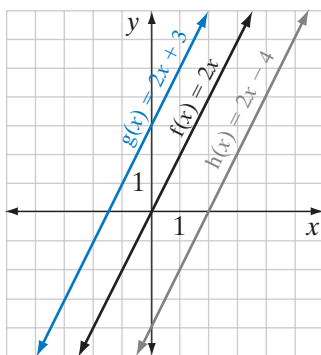
Compare the graphs of $f(x) = 2x$, $g(x) = 2x + 3$, and $h(x) = 2x - 4$.

1. $g(x) = f(x) + 3$. The graph of g is the graph of f shifted 3 units up.

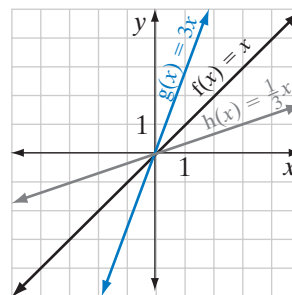**2.** $h(x) = f(x) - 4$. The graph of g is the graph of f shifted 4 units down.

Compare the graphs of $f(x) = 2x$, $g(x) = 2(x + 1)$, and $h(x) = 2(x - 2)$.

1. $g(x) = f(x + 1)$. The graph of g is the graph of f shifted 1 unit to the left in the x direction.

2. $h(x) = f(x - 2)$. The graph of g is the graph of f shifted 2 units to the right in the x direction.
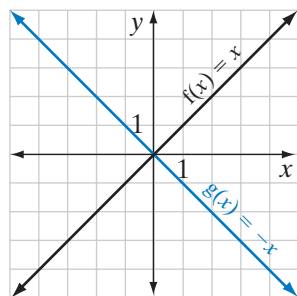
Compare the graphs of $f(x) = x$ and $g(x) = -x$.

- $g(x) = -f(x)$. The graph of g is the graph of f reflected in the x-axis.

Compare the graphs of $f(x) = x$, $g(x) = 3x$, and $h(x) = \frac{1}{3}x$.

1. $g(x) = 3f(x)$. The graph of g is the graph of f stretched vertically by a factor of 3.

2. $h(x) = \frac{1}{3}f(x)$. The graph of h is the graph of f compressed vertically by a factor of $\frac{1}{3}$.

Recall that in general, if $f(x)$ is a linear function, then:

▶ **The graph of $f(x) + a$ is the graph of $f(x)$ moved $|a|$ units up when a is positive and $|a|$ units down when a is negative.**

▶ **The graph of $f(x + a)$ is the graph of $f(x)$ moved $|a|$ units to the left when a is positive and $|a|$ units to the right when a is negative.**

▶ **The graph of $-f(x)$ is the graph of $f(x)$ reflected in the x-axis.**

▶ **When $a > 1$, the graph of $af(x)$ is the graph of $f(x)$ stretched vertically by a factor of a.**

▶ **When $0 < a < 1$, the graph of $af(x)$ is the graph of $f(x)$ compressed vertically by a factor of a.**

Direct Variation

If a car moves at a uniform rate of speed, the ratio of the distance that the car travels to the time that it travels is a constant. For example, when the car moves at 45 miles per hour,

$$\frac{d}{t} = 45$$

where d is distance in miles and t is time in hours.

The equation $\frac{d}{t} = 45$ can also be written as the function $\{(t, d) : d = 45t\}$. This function is a linear function whose domain and range are the set of real numbers. The function is one-to-one and onto. Although the function $d = 45t$

can have a domain and range that are the set of real numbers, when t represents the time traveled, the domain must be the set of positive real numbers. If the car travels no more than 10 hours a day, the domain would be $0 \le t \le 10$ and the range would be $0 \le d \le 450$.

When the ratio of two variables is a constant, we say that the variables are **directly proportional** or that the variables **vary directly**. Two variables that are directly proportional increase or decrease by the same factor. Every **direct variation** of two variables is a linear function that is one-to-one.

EXAMPLE 3

Jacob can type 55 words per minute.

a. Write a function that shows the relationship between the number of words typed, w, and the number of minutes spent typing, m.

b. Is the function one-to-one?

c. If Jacob types for no more than 2 hours at a time, what are the domain and range of the function?

Solution **a.** "Words per minute" can be written as $\frac{\text{words}}{\text{minutes}} = \frac{w}{m} = 55$. This is a direct variation function that can be written as $\{(m, w) : w = 55m\}$.

b. The function is a linear function and every linear function is one-to-one.

c. If Jacob types no more than 2 hours, that is, 120 minutes, at a time, the domain is $0 \le m \le 120$ and the range is $0 \le w \le 6{,}600$.

Exercises

Writing About Mathematics

1. Megan said that if $a > 1$ and $g(x) = \frac{1}{a}f(x)$, then the graph of $f(x)$ is the graph of $g(x)$ stretched vertically by the factor a. Do you agree with Megan? Explain why or why not.

2. Kyle said that if r is directly proportional to s, then there is some non-zero constant, c, such that $r = cs$ and that $\{(s, r)\}$ is a one-to-one function. Do you agree with Kyle? Explain why or why not.
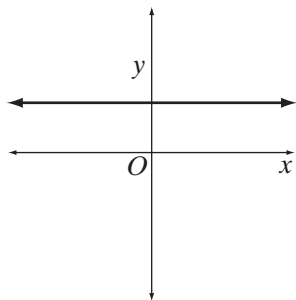
Developing Skills

In 3–6, each set represents a function. **a.** What is the domain of each function? **b.** What is the range of each function? **c.** Is the function one-to-one?
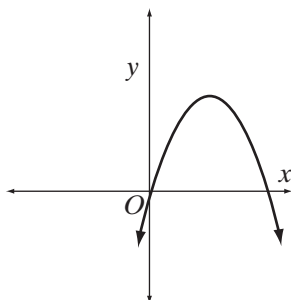
3. $\{(1, 4), (2, 7), (3, 10), (4, 13)\}$ **4.** $\{(0, 8), (2, 6), (4, 4), (6, 2)\}$

5. $\{(2, 7), (3, 7), (4, 7), (5, 7), (6, 7)\}$ **6.** $\{(0, 3), (-1, 5), (-2, 7), (-3, 9), (-4, 11)\}$

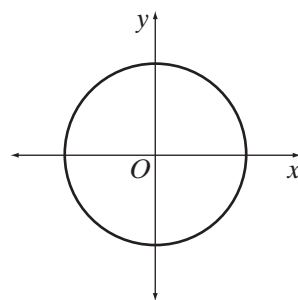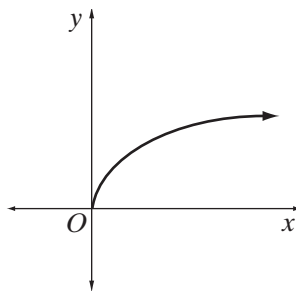In 7–12, determine if each graph represents a one-to-one function.
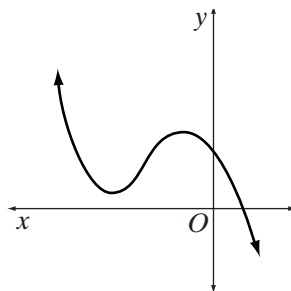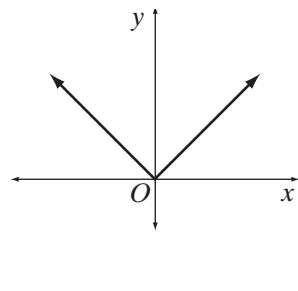
7.

8.

9.

10.

11.

12.

In 13–20: **a.** Graph each function. **b.** Is the function a direct variation? **c.** Is the function one-to-one?

13. $y = 3x$

14. $y = 6 - x$

15. $y = -x$

16. $y = \frac{8}{x}$

17. $y = \frac{1}{2}x$

18. $y = \sqrt{x}$

19. $\frac{y}{x} = 2$

20. $\frac{x}{y} = 2$

21. Is every linear function a direct variation?

22. Is the direct variation of two variables always a linear function?

Applying Skills

In 23–28, write an equation of the direct variation described.

23. The cost of tickets, c, is directly proportional to the number of tickets purchased, n. One ticket costs six dollars.

24. The distance in miles, d, that Mr. Spencer travels is directly proportional to the length of time in hours, t, that he travels at 35 miles per hour.

25. The length of a line segment in inches, i, is directly proportional to the length of the segment in feet, f.

26. The weight of a package of meat in grams, g, is directly proportional to the weight of the package in kilograms, k.

27. Water is flowing into a swimming pool at the rate of 25 gallons per minute. The number of gallons of water in the pool, g, is directly proportional to the number of minutes, m, that the pool has been filling from when it was empty.

28. At 9:00 A.M., Christina began to add water to a swimming pool at the rate of 25 gallons per minute. When she began, the pool contained 80 gallons of water. Christina stopped adding water to the pool at 4:00 P.M. Let g be the number of gallons of water in the pool and t be the number of minutes that have past since 9:00 A.M.

   a. Write an equation for g as a function of t.

   b. What is the domain of the function?

   c. What is the range of the function?

   d. Is the function one-to-one?

   e. Is the function an example of direct variation? Explain why or why not.

## Hands-On Activity 1

Refer to the graph of f shown to the right.

   1. Sketch the graph of $g(x) = f(-x)$.

   2. Sketch the graph of $h(x) = -f(x)$.

   3. Describe the relationship between the graph of $f(x)$ and the graph of $g(x)$.

   4. Describe the relationship between the graph of $f(x)$ and the graph of $h(x)$.

   5. Sketch the graph of $p(x) = 3f(x)$.

   6. Describe the relationship between the graph of $f(x)$ and the graph of $p(x)$.

   7. Describe the relationship between the graph of $f(x)$ and the graph of $-p(x)$.

   8. Sketch the graph of $f(x) = x + 1$ and repeat steps 1 through 7.
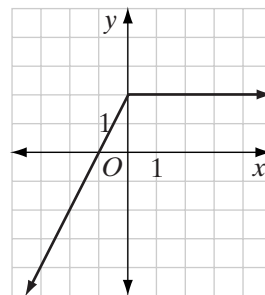
Hands-On Activity 2

Let f be the function whose graph is shown to the right.

   1. Sketch the graph of $g(x) = f(x + 2)$.

   2. Sketch the graph of $h(x) = f(x - 4)$.

   3. Describe the relationship between the graph of f and the graph of $p(x) = f(x + a)$ when a is positive.

   4. Describe the relationship between the graph of f and the graph of $p(x) = f(x + a)$ when a is negative.

   5. Sketch the graph of $f(x) + 2$.

   6. Sketch the graph of $f(x) - 4$.

   7. Describe the relationship between the graph of f and the graph of $f(x) + a$ when a is positive.

   8. Describe the relationship between the graph of f and the graph of $f(x) + a$ when a is negative.
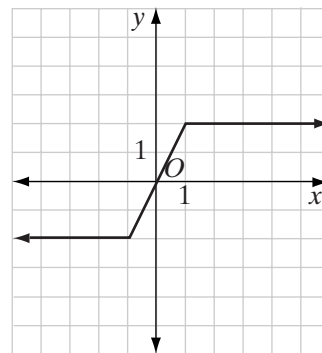
9. Sketch the graph of $2f(x)$.

10. Describe the relationship between $f(x)$ and $af(x)$ when a is a constant greater than 0.

11. Sketch the graph of $-f(x)$.

12. Describe the relationship between $f(x)$ and $-f(x)$.

13. Sketch the graph of $g(x) = f(-x)$.

14. Describe the relationship between $f(x)$ and $g(x)$.

4-4 ABSOLUTE VALUE FUNCTIONS

Let $f(x) = |x|$ be an **absolute value function** from the set of real numbers to the set of real numbers. When the domain is the set of real numbers, the range of f is the set of non-negative real numbers. Therefore, the absolute value function is not onto. For every positive number a, there are two ordered pairs (a, a) and $(-a, a)$ that are pairs of the function. The function is not one-to-one. We say that the function is **many-to-one** because many (in this case two) different first elements, a and $-a$, have the same second element a.
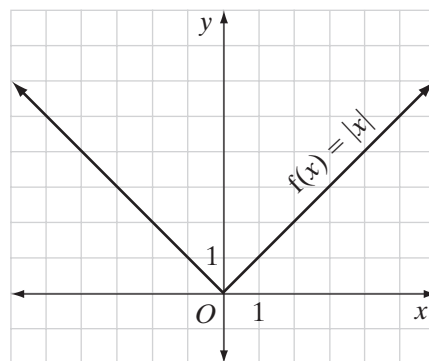
We use single letters such as f and g to name a function. For special functions three letters are often used to designate the function. For example, on the graphing calculator, the absolute value function is named abs. The absolute value function, abs(, is located in the MATH menu under the NUM heading. For instance, to evaluate $|2(1) - 5|$:

ENTER: | MATH | ▶ | 1 | 2 | (| 1 |) | − | 5 |) | ENTER |

DISPLAY:
```
abs(2(1)-5)
              3
```

Note that the opening parenthesis is supplied by the calculator. We enter the closing parenthesis after we have entered the expression enclosed between the absolute value bars.
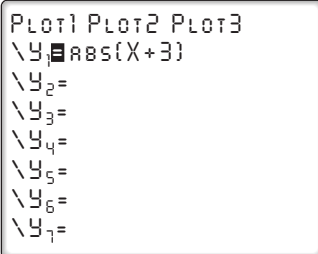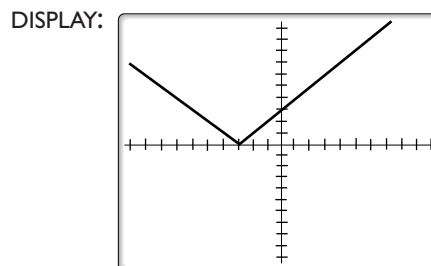
EXAMPLE 1

Graph the function $y = |x + 3|$ on a graphing calculator.

Solution To display the graph of $y = |x + 3|$, first enter the function value into Y_1. Then graph the function in the ZStandard window.

ENTER: Y=   MATH   ▶   1   3   ENTER: ZOOM   6

X,T,Θ,*n*   +   3   )

DISPLAY:
```
Plot1 Plot2 Plot3
\Y₁∎abs(X+3)
\Y₂=
\Y₃=
\Y₄=
\Y₅=
\Y₆=
\Y₇=
```

DISPLAY:

Note that the minimum value of y is 0. The x value when $y = 0$ is $x + 3 = 0$ or $x = -3$.

The graph of an absolute value function can be used to find the roots of an absolute value equation or inequality.
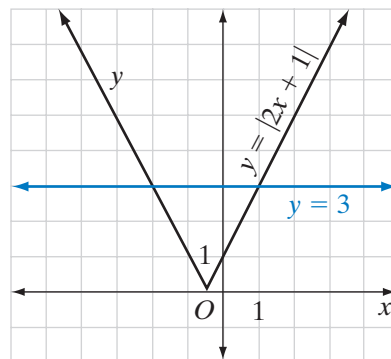
EXAMPLE 2

Use a graph to find the solution set of the following:

a. $|2x + 1| = 3$ **b.** $|2x + 1| < 3$ **c.** $|2x + 1| > 3$.

Solution Draw the graph of $y = |2x + 1|$.

On the same set of axes, draw the graph of $y = 3$.

a. The points of intersection of the graphs are $(-2, 3)$ and $(1, 3)$, so $|2x + 1| = 3$ for $x = -2$ or $x = 1$.

b. Between $x = -2$ and $x = 1$, the graph of $y = |2x + 1|$ lies below the graph of $y = 3$, so $|2x + 1| < 3$ for $-2 < x < 1$.

c. For $x < -2$ and $x > 1$, the graph of $y = |2x + 1|$ lies above the graph of $y = 3$, so $|2x + 1| > 3$ for $x < -2$ or $x > 1$.

Answers **a.** $\{-2, 1\}$ **b.** $\{x : -2 < x < 1\}$ **c.** $\{x : x < -2 \text{ or } x > 1\}$

EXAMPLE 3

Find the coordinates of the ordered pair of the function $y = |4x - 2|$ for which y has a minimum value.

Solution Since $y = |4x - 2|$ is always non-negative, its minimum y-value is 0.

When $|4x - 2| = 0$, $4x - 2 = 0$. Solve this linear equation for x:

$$4x - 2 = 0$$
$$4x = 2$$
$$x = \tfrac{2}{4} = \tfrac{1}{2}$$

Calculator Solution Enter the function $y = |4x - 2|$ into Y_1 and then press **GRAPH**. Press **2nd** **CALC** to view the CALCULATE menu and choose 2: zero. Use the arrows to enter a left bound to the left of the minimum, a right bound to the right of the minimum, and a guess near the minimum. The calculator will display the coordinates of the minimum, which is where the graph intersects the y-axis.
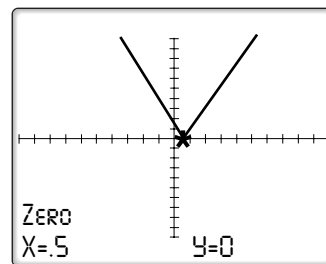
Answer The minimum occurs at $\left(\tfrac{1}{2}, 0\right)$.

Exercises

Writing About Mathematics

1. If the domain of the function $f(x) = |3 - x|$ is the set of real numbers less than 3, is the function one-to-one? Explain why or why not.

2. Eric said that if $f(x) = |2 - x|$, then $y = 2 - x$ when $x \leq 2$ and $y = x - 2$ when $x > 2$. Do you agree with Eric? Explain why or why not.

Developing Skills

In 3–6, find the coordinates of the ordered pair with the smallest value of y for each function.

3. $y = |x|$ **4.** $y = |2x + 8|$ **5.** $f(x) = \left|\tfrac{x}{2} - 7\right|$ **6.** $g(x) = |5 - x|$

In 7–10, the domain of each function is the set of real numbers. **a.** Sketch the graph of each function. **b.** What is the range of each function?

7. $y = |x - 1|$ **8.** $y = |3x + 9|$ **9.** $f(x) = |4x| + 1$ **10.** $g(x) = \left|3 - \tfrac{x}{2}\right| - 3$

Applying Skills

11. $A(2, 7)$ is a fixed point in the coordinate plane. Let $B(x, 7)$ be any point on the same horizontal line. If $AB = $ h(x), express h(x) in terms of x.

12. Along the New York State Thruway there are distance markers that give the number of miles from the beginning of the thruway. Brianna enters the thruway at distance marker 150.

    a. If m$(x) = $ the number of miles that Brianna has traveled on the thruway, write an equation for m(x) when she is at distance marker x.

    b. Write an equation for h(x), the number of hours Brianna required to travel to distance marker x at 65 miles per hour

13. a. Sketch the graph of $y = |x|$.

    b. Sketch the graph of $y = |x| + 2$

    c. Sketch the graph of $y = |x| - 2$.

    d. Describe the graph of $y = |x| + a$ in terms of the graph of $y = |x|$.

14. a. Sketch the graph of $y = |x|$.

    b. Sketch the graph of $y = |x + 2|$.

    c. Sketch the graph of $y = |x - 2|$.

    d. Describe the graph of $y = |x + a|$ in terms of the graph of $y = |x|$.

15. a. Sketch the graph of $y = |x|$.

    b. Sketch the graph of $y = -|x|$.

    c. Describe the graph of $y = -|x|$ in terms of the graph of $y = |x|$.

16. a. Sketch the graph of $y = |x|$.

    b. Sketch the graph of $y = 2|x|$.

    c. Sketch the graph of $y = \frac{1}{2}|x|$.

    d. Describe the graph of $y = a|x|$ in terms of the graph of $y = |x|$.

17. a. Draw the graphs of $y = |x + 3|$ and $y = 5$.

    b. From the graph drawn in **a**, determine the solution set of $|x + 3| = 5$.

    c. From the graph drawn in **a**, determine the solution set of $|x + 3| > 5$.

    d. From the graph drawn in **a**, determine the solution set of $|x + 3| < 5$.

18. a. Draw the graphs of $y = -|x - 4|$ and $y = -2$.

    b. From the graph drawn in **a**, determine the solution set of $-|x - 4| = -2$.

    c. From the graph drawn in **a**, determine the solution set of $-|x - 4| > -2$.

    d. From the graph drawn in **a**, determine the solution set of $-|x - 4| < -2$.

4-5 POLYNOMIAL FUNCTIONS

Recall that a monomial is a constant, a variable, or the product of constants and variables. For example, $5x$ is a monomial in x. A polynomial is a monomial or the sum of monomials. The expression $x^3 - 5x^2 - 4x + 20$ is a polynomial in x.

A function f is a *polynomial function* when f(x) is a polynomial in x. For example, f(x) $= x^3 - 5x^2 - 4x + 20$ is a polynomial function.

DEFINITION

A **polynomial function f of degree n** is a function of the form
$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0, a_n \neq 0.$$

The simplest polynomial function is the constant function, $y = $ a constant. For example, $y = 5$ is a constant function whose graph is a horizontal line.

The linear function is a polynomial function of degree one. When $m \neq 0$, the graph of the linear function $y = mx + b$ is a slanted line. The y-intercept of the line is b and the slope of the line is m.

Quadratic Functions

The **quadratic function** is a polynomial function of degree two. When $a \neq 0$, the graph of the quadratic function $y = ax^2 + bx + c$ is a curve called a **parabola**.

When a is positive, the parabola opens upward and has a minimum value of y. When $a = 1, b = 2$, and $c = -3$, the quadratic function is $y = x^2 + 2x - 3$.
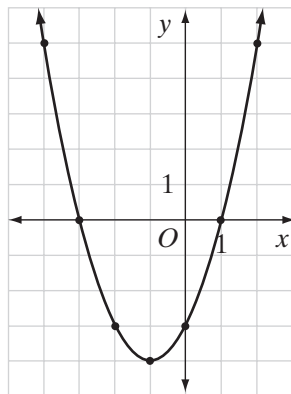
x	$x^2 + 2x - 3$	y
-4	$(-4)^2 + 2(-4) - 3$	5
-3	$(-3)^2 + 2(-3) - 3$	0
-2	$(-2)^2 + 2(-2) - 3$	-3
-1	$(-1)^2 + 2(-1) - 3$	-4
0	$(0)^2 + 2(0) - 3$	-3
1	$(1)^2 + 2(1) - 3$	0
2	$(2)^2 + 2(2) - 3$	5

The **turning point**, also called the **vertex**, for this parabola is $(-1, -4)$ and the **axis of symmetry of the parabola** is the line $x = -1$, the vertical line through the turning point.

When a is negative, the parabola opens downward and has a maximum value of y. When $a = -1, b = -2$, and $c = 3$, the quadratic function is $y = -x^2 - 2x + 3$.
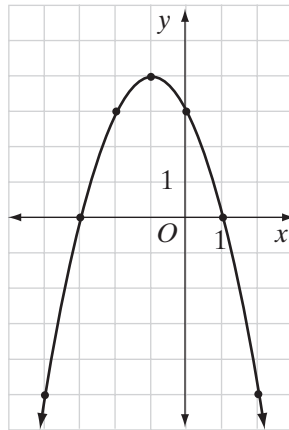
x	$-x^2 - 2x + 3$	y
-4	$-(-4)^2 - 2(-4) + 3$	-5
-3	$-(-3)^2 - 2(-3) + 3$	0
-2	$-(-2)^2 - 2(-2) + 3$	3
-1	$-(-1)^2 - 2(-1) + 3$	4
0	$-(0)^2 - 2(0) + 3$	3
1	$-(1)^2 - 2(1) + 3$	0
2	$-(2)^2 - 2(2) + 3$	-5

The turning point for this parabola is $(-1, 4)$ and the axis of symmetry is the line $x = -1$, the vertical line through the turning point.

Recall from previous courses that for the parabola that is the graph of $f(x) = ax^2 + bx + c$:

1. The minimum or maximum value of the parabola occurs at the axis of symmetry.

2. The formula for the axis of symmetry of the parabola is:

$$x = \frac{-b}{2a}$$

Transformations of Quadratic Functions

The quadratic function $g(x) = x^2 + 6x + 10$ can be written as $g(x) = (x + 3)^2 + 1$.

If: $f(x) = x^2$

$g(x) = (x + 3)^2 + 1$

Then: $g(x) = f(x + 3) + 1$

The function $f(x + 3) + 1$ moves every point of $f(x)$ 3 units to the left and 1 unit up. In particular, since the vertex of f is $(0, 0)$, the vertex of g is $(-3, 1)$. Note that for every point on f, there is a corresponding point on g that is 3 units to the left and 1 unit up as shown in the graph on the right.
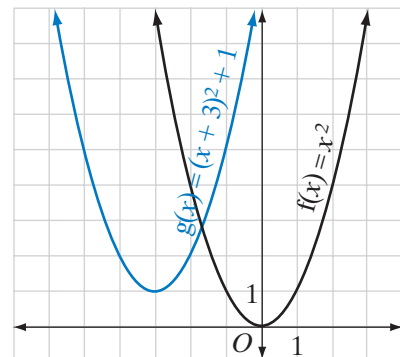
How does the graph of $g(x) = 2f(x)$ and the graph of $h(x) = \frac{1}{2}f(x)$ compare with the graph of $f(x)$? The graphs of $f(x) = x^2$, $g(x) = 2x^2$, and $h(x) = \frac{1}{2}x^2$ are shown to the right. Compare points on $f(x)$, $g(x)$, and $h(x)$ that have the same x-coordinate. The graph of g is the graph of f stretched vertically by a factor of 2. In other words, the y-value of a point on the graph of g is twice the y value of the point on f and the y value of a point on h is one-half the y value of the point on f.
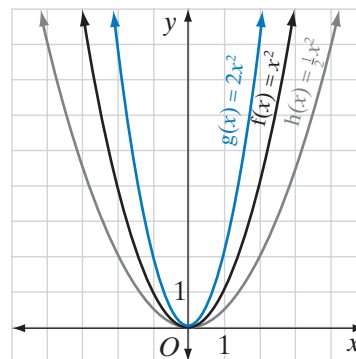

In general, if $f(x)$ is a quadratic function, then:

▶ **The vertex of a quadratic function of the form $f(x) = a(x - h)^2 + k$ is (h, k).**

▶ **The function $g(x) = f(x - h) + k$ moves every point of $f(x)$ $|h|$ units to the left if h is positive and to the right if h is negative and $|k|$ units up if k is positive and down if k is negative.**

▶ **When $f(x)$ is multiplied by a when $a > 1$, the graph is stretched in the vertical direction and when $f(x)$ is multiplied by a when $0 < a < 1$, $f(x)$ is compressed in the vertical direction.**

EXAMPLE 1

Graph $y = \frac{1}{3}(x - 4)^2 + 3$.

Solution The graph of $y = \frac{1}{3}(x - 4)^2 + 3$ is the graph of $y = x^2$ compressed vertically by a factor of $\frac{1}{3}$, shifted to the right by 4 units and up by 3 units. The vertex of the parabola is $(4, 3)$ and it opens upward as shown in the graph.
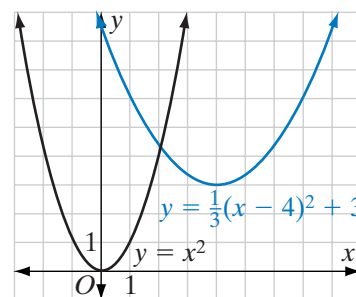
Roots of a Polynomial Function

The **roots** or **zeros** of a polynomial function are the values of x for which $y = 0$. These are the x-coordinates of the points where the graph of the function intersects the x-axis. For the function $y = x^2 + 2x - 3$ as well as for the function $y = -x^2 - 2x + 3$, the roots or zeros of the function are -3 and 1.

These roots can also be found by letting $y = 0$ and solving the resulting quadratic functions for x.

$$y = x^2 + 2x - 3 \qquad\qquad y = -x^2 - 2x + 3$$
$$0 = x^2 + 2x - 3 \qquad\qquad 0 = -(x^2 + 2x - 3)$$
$$0 = (x + 3)(x - 1) \qquad\qquad 0 = -(x + 3)(x - 1)$$

$x + 3 = 0$	$x - 1 = 0$	$x + 3 = 0$	$x - 1 = 0$
$x = -3$	$x = 1$	$x = -3$	$x = 1$

Higher Degree Polynomial Functions

A polynomial function of degree two, a quadratic function, has one turning point. A polynomial function of degree three has at most two turning points. The graph at the right is the graph of the polynomial function $y = x^3 + x^2 - x - 1$. This graph has a turning point at $x = -1$ and at a point between $x = 0$ and $x = 1$.

A polynomial function of degree three can have one or three real roots or zeros. It appears that the polynomial function $y = x^3 + x^2 - x - 1$ has only two real roots. However, the root -1, at which the graph is tangent to the x-axis, is called a **double root**, a root that occurs twice. We can see that this is true by solving the polynomial function algebraically. To find the roots of a polynomial function, let y equal zero.
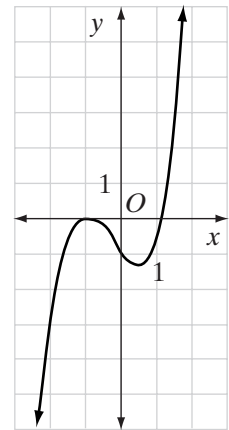
$$y = x^3 + x^2 - x - 1$$
$$0 = x^3 + x^2 - x - 1$$
$$0 = x^2(x + 1) - 1(x + 1)$$
$$0 = (x + 1)(x^2 - 1)$$
$$0 = (x + 1)(x + 1)(x - 1)$$

$x + 1 = 0$	$x + 1 = 0$	$x - 1 = 0$
$x = -1$	$x = -1$	$x = 1$

The polynomial function of degree three has three real roots, but two of these roots are equal. Note that if 1 is a root of the polynomial function, $(x - 1)$ is a factor of the polynomial, and if -1 is a double root of the polynomial function, $(x + 1)(x + 1)$ or $(x + 1)^2$ is a factor of the polynomial. In general:

▶ **If a is a root of a polynomial function, then $(x - a)$ is a factor of the polynomial.**

Note also that the polynomial function $y = x^3 + x^2 - x - 1$ has three roots. Any polynomial function of degree three has 1 or 3 real roots. If it has two real roots, one of them is a double root. These real roots may be irrational numbers. In general:

▶ **A polynomial function of degree *n* has at most *n* distinct real roots.**

Examining the Roots of Polynomial from the Graph

We found the roots of the polynomial $y = x^3 + x^2 - x - 1$ by factoring, but not all polynomials can be factored over the set of integers. However, we can use a calculator to sketch the graph of the function and examine the roots using the zero function.

For example, to examine the roots of $f(x) = x^4 - 4x^3 - x^2 + 16x - 12$, enter the function into Y_1 and view the graph in the ZStandard window. The graph appears to cross the *x*-axis at -2, 1, 2, and 3. We can verify that these are the roots of the function.

Press $\boxed{\textbf{2nd}}$ $\boxed{\textbf{CALC}}$ to view the CALCULATE menu and choose 2: zero. Use the arrows to enter a left bound to the left of one of the zero values, a right bound to the right of the zero value, and a guess near the zero value. The calculator will display the coordinates of the point at which the graph intersects the *x*-axis. Repeat to find the other roots.
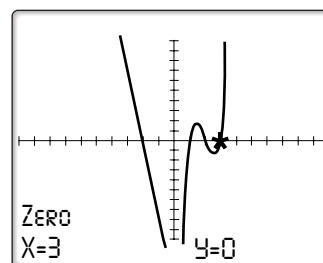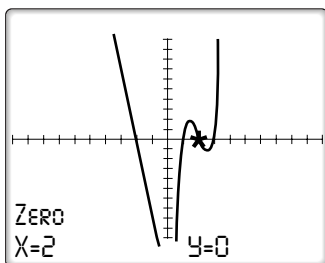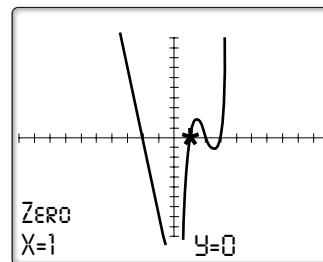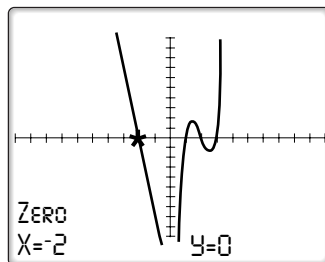
We can verify that these are the roots of the function by evaluating the function at each value:

$$f(-2) = (-2)^4 - 4(-2)^3 - (-2)^2 + 16(-2) - 12$$
$$= 16 - 4(-8) - 4 - 32 - 12$$
$$= 0$$

$$f(2) = (2)^4 - 4(2)^3 - (2)^2 + 16(2) - 12$$
$$= 16 - 4(8) - 4 + 32 - 12$$
$$= 0$$

$$f(1) = (1)^4 - 4(1)^3 - (1)^2 + 16(1) - 12$$
$$= 1 - 4 - 1 + 16 - 12$$
$$= 0$$

$$f(3) = (3)^4 - 4(3)^3 - (3)^2 + 16(3) - 12$$
$$= 81 - 4(27) - 9 + 48 - 12$$
$$= 0$$

Note: The calculator approach gives exact values only when the root is an integer or a rational number that can be expressed as a finite decimal. The calculator returns approximate values for rational roots with infinite decimal values and for irrational roots.

EXAMPLE 2

From the graph, determine the roots of the polynomial function p(x).
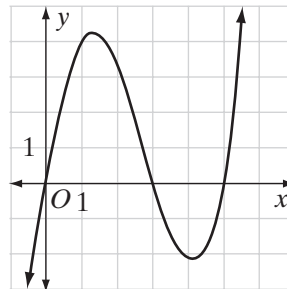
Solution   The graph intersects the *x*-axis at 0, 3, and 5.

Therefore, the roots or zeros of the function are 0, 3, and 5.

EXAMPLE 3

Find the real roots of $f(x) = x^4 - 2x^3 - 2x^2 - 2x - 3$ graphically.

Solution   Enter the function into Y_1 and graph the func-
tion into the standard viewing window. The
graph appears to cross the x-axis at 3 and -1.
We can verify that these are the roots of the
function. Press **2nd** **CALC** **2** to use the
zero function. Use the arrows to enter a left
bound to the left of one of the zero values, a
right bound to the right of the zero value, and a
guess near the zero value. Examine one zero
value at a time. Repeat to find the other roots.

```
ZERO
X=-1            Y=0
```

```
ZERO
X=3             Y=0
```
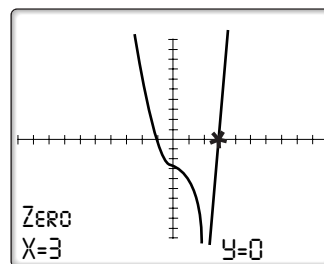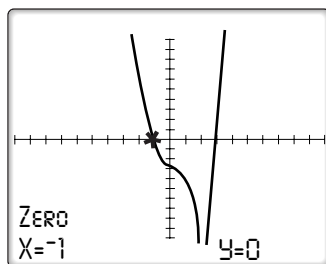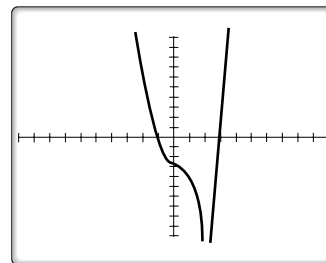
Answer   The real roots of the function are -1 and 3.

EXAMPLE 4

Approximate the real roots of
$f(x) = \frac{1}{3}(x^3 + x^2 - 11x - 3)$ to the
nearest tenth.

Solution   The graph appears to cross the x-
axis at -3.7, -0.3, and 3. We can
verify that these are valid approxi-
mations of the roots of the function
by using the graphing calculator.
Enter the function into Y_1 and
graph the function into the
standard viewing window. Press
2nd **CALC** **2** to use the zero
function. Use the arrows to enter a left bound to the left of one of the zero
values, a right bound to the right of the zero value, and a guess near the zero
value. Examine one zero value at a time. Repeat to find the other roots.
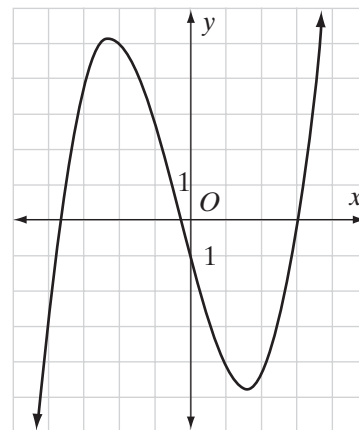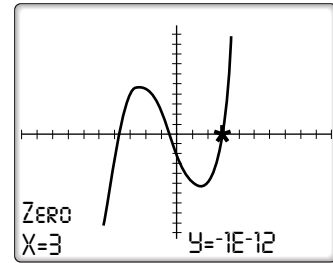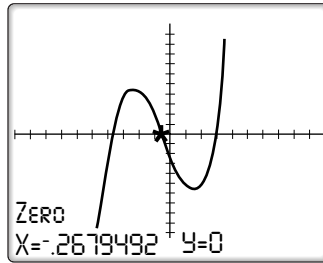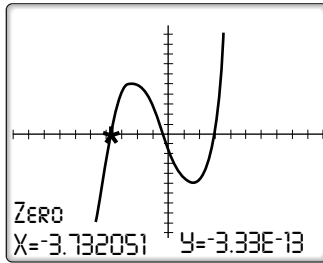
Answer The real roots of the function are approximately equal to -3.73, -0.27, and 3.

Exercises

Writing About Mathematics

1. Tiffany said that the polynomial function $f(x) = x^4 + x^2 + 1$ cannot have real roots. Do you agree with Tiffany? Explain why or why not.

2. The graph of $y = x^2 - 4x + 4$ is tangent to the x-axis at $x = 2$ and does not intersect the x-axis at any other point. How many roots does this function have? Explain your answer.

Developing Skills

In 3–11, each graph shows a polynomial functions from the set of real numbers to the set of real numbers.

a. Find the real roots or zeros of the function from the graph, if they exist.

b. What is the range of the function?

c. Is the function onto?

d. Is the function one-to-one?

3.

4.

5.

6.

7.

8.

9.

10.

11.
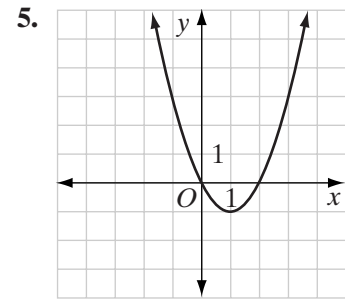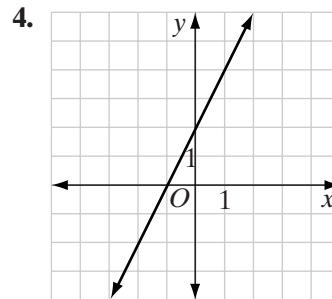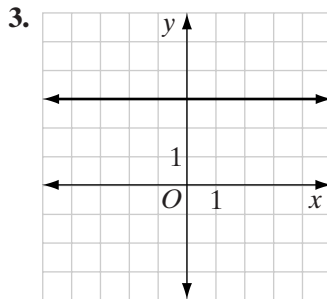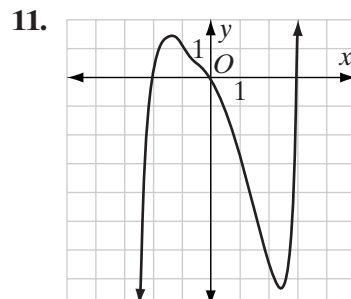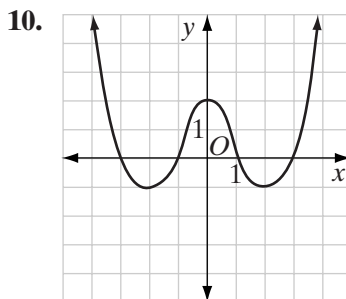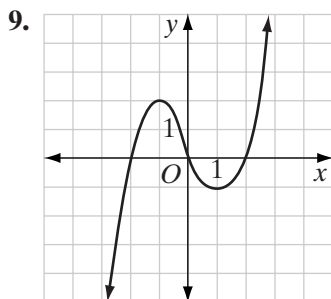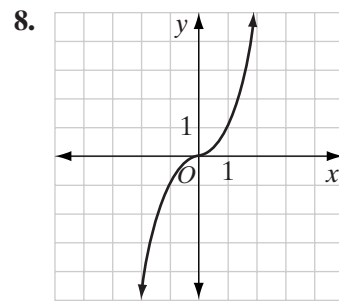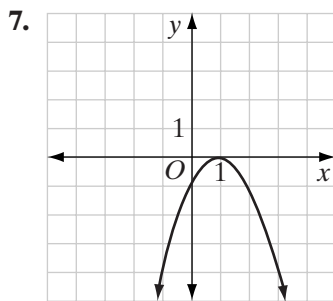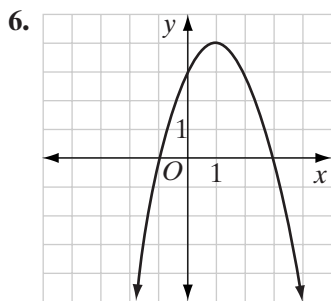
In 12–17, use a graph to find the solution set of each inequality.

12. $x^2 + 2x - 3 < 0$ **13.** $x^2 + 4x + 3 \le 0$ **14.** $x^2 - x - 2 > 0$

15. $x^2 - 2x + 1 < 0$ **16.** $-x^2 + 6x - 5 < 0$ **17.** $-x^2 - 3x + 4 \ge 0$

Applying Skills

18. The perimeter of a rectangle is 24 feet.

     a. If x is the measure of one side of the rectangle, represent the measure of an adjacent side in terms of x.

     b. If y is the area of the rectangle, express the area in terms of x.

     c. Draw the graph of the function written in **b**.

     d. What are the dimensions of the rectangle with the largest area?

19. The sum of the lengths of the legs of a right triangle is 20 feet.

     a. If x is the measure of one of the legs, represent the measure of the other leg in terms of x.

     b. If y is the area of the triangle, express the area in terms of x.

     c. Draw the graph of the function written in **b**.

     d. What are the dimensions of the triangle with the largest area?

20. A polynomial function of degree three, p(x), intersects the x-axis at $(-4, 0)$, $(-2, 0)$, and $(3, 0)$ and intersects the y-axis at $(0, -24)$. Find p(x).

21. a. Sketch the graph of $y = x^2$.

   b. Sketch the graph of $y = x^2 + 2$

   c. Sketch the graph of $y = x^2 - 3$.

   d. Describe the graph of $y = x^2 + a$ in terms of the graph of $y = x^2$.

   e. What transformation maps $y = x^2$ to $y = x^2 + a$?

22. a. Sketch the graph of $y = x^2$.

   b. Sketch the graph of $y = (x + 2)^2$.

   c. Sketch the graph of $y = (x - 3)^2$.

   d. Describe the graph of $y = (x + a)^2$ in terms of the graph of $y = x^2$.

   e. What transformation maps $y = x^2$ to $y = (x + a)^2$?

23. a. Sketch the graph of $y = x^2$.

   b. Sketch the graph of $y = -x^2$.

   c. Describe the graph of $y = -x^2$ in terms of the graph of $y = x^2$.

   d. What transformation maps $y = x^2$ to $y = -x^2$?

24. a. Sketch the graph of $y = x^2$.

   b. Sketch the graph of $y = 3x^2$

   c. Sketch the graph of $y = \frac{1}{3}x^2$.

   d. Describe the graph of $y = ax^2$ in terms of the graph of $y = x^2$ when $a > 1$.

   e. Describe the graph of $y = ax^2$ in terms of the graph of $y = x^2$ when $0 < a < 1$.

25. For the parabola whose equation is $y = ax^2 + bx + c$, the equation of the axis of symmetry is $x = \frac{-b}{2a}$. The turning point of the parabola lies on the axis of symmetry. Therefore its x-coordinate is $\frac{-b}{2a}$. Substitute this value of x in the equation of the parabola to find the y-coordinates of the turning point. Write the coordinates of the turning point in terms of a, b, and c.

4-6 THE ALGEBRA OF FUNCTIONS

Function Arithmetic

Fran and Greg work together to produce decorative vases. Fran uses a potter's wheel to form the vases and Greg glazes and fires them. Last week, they completed five jobs. The table on the following page shows the number of hours spent on each job.

Job number	Hours Worked		
	Fran	**Greg**	**Total**
1	3	7	10
2	2	5	7
3	4	8	12
4	$\frac{1}{2}$	3	$3\frac{1}{2}$
5	$1\frac{1}{2}$	6	$7\frac{1}{2}$

Let x equal the job number, $f(x) =$ the number of hours that Fran worked on job x, $g(x) =$ the number of hours that Greg worked on job x, and $t(x) =$ the total number of hours worked on job x.

Therefore:

$$
\begin{array}{lll}
f(1) = 3 & g(1) = 7 & t(1) = 10 \\
f(2) = 2 & g(2) = 5 & t(2) = 7 \\
f(3) = 4 & g(3) = 8 & t(3) = 12 \\
f(4) = \frac{1}{2} & g(4) = 3 & t(4) = 3\frac{1}{2} \\
f(5) = 1\frac{1}{2} & g(5) = 6 & t(5) = 7\frac{1}{2}
\end{array}
$$

For each function, the domain is the set $\{1, 2, 3, 4, 5\}$ and $t(x) = f(x) + g(x)$ or $t(x) = (f + g)(x)$. We can perform arithmetic operations with functions. In general, if $f(x)$ and $g(x)$ are functions, then:

$$\textbf{(f + g)(x) = f(x) + g(x)}$$

The domain of $(f + g)$ is the set of elements common to the domain of f and of g, that is, the domain of $(f + g)$ is the intersection of the domains of f and g.

For example, if $f(x) = 2x^2$ and $g(x) = 5x$, then:

$$(f + g)(x) = f(x) + g(x) = 2x^2 + 5x$$

If the domains of f and g are both the set of real numbers, then the domain of $(f + g)$ is also the set of real numbers.

The difference, product, and quotient of two functions, as well as the product of a constant times a function, can be defined in a similar way.

If $f(x)$ and $g(x)$ are functions, then:

1. The difference of f and g is expressed as:

$$\textbf{(f − g)(x) = f(x) − g(x)}$$

Example: Let $f(x) = 2x^2$ and $g(x) = 5x$.
Then $(f − g)(x) = f(x) − g(x) = 2x^2 − 5x$.

2. The product of f and g is expressed as:

$$(fg)(x) = f(x) \cdot g(x)$$

Example: Let $f(x) = 2x^2$ and $g(x) = 5x$.
Then $(fg)(x) = f(x) \cdot g(x) = 2x^2(5x) = 10x^3$.

3. The quotient of f and g is expressed as:

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

Example: Let $f(x) = 2x^2$ and $g(x) = 5x$. Then $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{2x^2}{5x} = \frac{2x}{5}$.

4. The product of f and a constant, a, is expressed as:

$$af(x)$$

Example: Let $f(x) = 2x^2$ and $a = 3$. Then $3f(x) = 3(2x^2) = 6x^2$.

Note: If $g(x) = a$, a constant, then $(fg)(x) = f(x)g(x) = f(x) \cdot (a) = af(x)$. That is, if one of the functions is a constant, then function multiplication is the product of a function and a constant.

If the domain of f and of g is the set of real numbers, then the domain of $(f + g)$, $(f - g)$, (fg), and $af(x)$ is the set of real numbers and the domain of $\left(\frac{f}{g}\right)$ is the set of numbers for which $g(x) \neq 0$.

In the examples given above, the domain of f and of g is the set of real numbers. The domain of $f + g$, of $f - g$, and of fg is the set of real numbers. The domain of $\frac{f}{g}$ is the set of real numbers for which $g(x) \neq 0$, that is, the set of non-zero real numbers.

EXAMPLE 1

Let $f(x) = 3x$ and $g(x) = \sqrt{x}$.

a. Write an algebraic expression for $h(x)$ if $h(x) = 2f(x) \div g(x)$.

b. What is the largest possible domain for f, 2f, g, and h that is a subset of the set of real numbers?

c. What are the values of $f(2)$, $g(2)$, and $h(2)$?

Solution   **a.** $h(x) = \frac{2f(x)}{g(x)} = \frac{2(3x)}{\sqrt{x}} = \frac{6x}{\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}} = \frac{6x\sqrt{x}}{x} = 6\sqrt{x}$

b. The domain of f and of 2f is the set of real numbers.

The domain of g is the set of non-negative real numbers.

The domain of h is the set of positive real numbers, that is, the set of all numbers in the domains of both f and g for which $g(x) \neq 0$.

c. $f(2) = 3(2) = 6$ $g(2) = \sqrt{2}$ $h(2) = 6\sqrt{2}$

Transformations and Function Arithmetic

Let $h(x) = 3$ be a constant function. The second element of each ordered pair is the constant, 3. The domain of h is the set of real numbers and the range is {3}. The graph of $h(x) = 3$ is a horizontal line 3 units above the x-axis.

Let $g(x) = |x|$. The domain of g is the set of real numbers and the range is the set of non-negative real numbers.

Let $f(x) = g(x) + h(x) = |x| + 3$. The domain of $f(x)$ is the set of real numbers. For each value of x, $f(x)$ is 3 more than $g(x)$. The graph of $f(x)$ is the graph of $g(x)$ translated 3 units in the vertical direction.

In general, for any function g:

▶ **If $h(x) = a$, a constant, and $f(x) = g(x) + h(x) = g(x) + a$, then the graph of $f(x)$ is the graph of $g(x)$ translated a units in the vertical direction.**
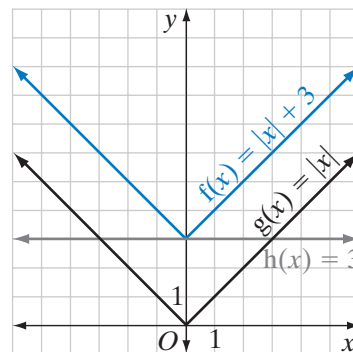
Let h and g be defined as before.

Let $f(x) = g(x) \cdot h(x) = gh(x) = 3|x|$. The domain of $f(x)$ is the set of real numbers. For each value of x, $f(x)$ is 3 times $g(x)$. The graph of $f(x)$ is the graph of $g(x)$ stretched by the factor 3 in the vertical direction.

In general, for any function g:

▶ **If $h(x) = a$, and a is a positive constant, then $f(x) = g(x) \cdot h(x) = ag(x)$. The graph of $f(x)$ is the graph of $g(x)$ stretched by the factor a in the vertical direction when $a > 1$ and compressed by a factor of a when $0 < a < 1$.**
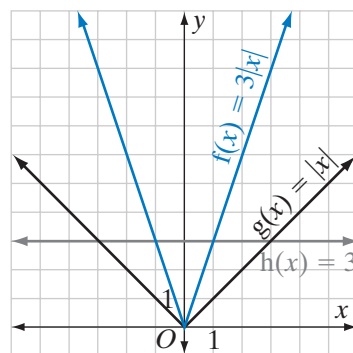
Let $h(x) = -1$ be a constant function. The second element of each pair is the constant, -1. The domain of h is the set of real numbers and the range is {-1}.

Let $g(x) = |x|$. The domain of g is the set of real numbers and the range is the set of non-negative real numbers.
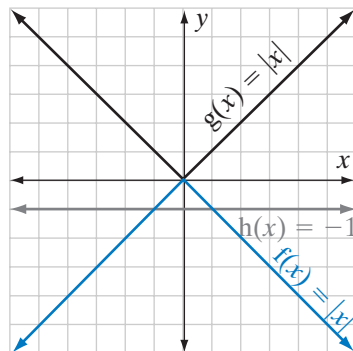
Let $f(x) = g(x) \cdot h(x) = gh(x) = -1|x| = -|x|$. The domain of $f(x)$ is the set of real numbers. For each value of x, $f(x)$ is -1 times $g(x)$, that is, the opposite of $g(x)$. The graph of $f(x)$ is the graph of $g(x)$ reflected in the x-axis.

In general, for any function g:

▶ **If $h(x) = -1$, then $f(x) = g(x) \cdot h(x) = -g(x)$. The graph of $f(x)$ is the graph of $g(x)$ reflected in the x-axis.**

Horizontal translations will be covered in the next section when we discuss function composition.

EXAMPLE 2

The graphs of $f(x)$, $g(x)$, and $h(x)$ are shown at the right. The function $g(x)$ is the function $f(x)$ compressed vertically by the factor $\frac{1}{2}$, and the function $h(x)$ is the function $g(x)$ moved 3 units up.

If $f(x) = \sqrt{x}$, what is the equation of $h(x)$?
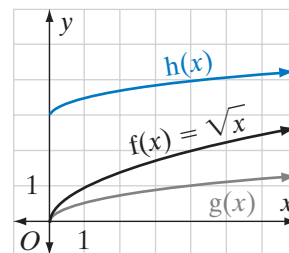
Solution The vertex of the parabola that is the graph of $f(x)$ is the point $(0, 0)$. When $f(x)$ is compressed by the factor $\frac{1}{2}$,

$$g(x) = \tfrac{1}{2}f(x) = \tfrac{1}{2}\sqrt{x}$$

Answer When $g(x)$ is moved 3 units up, $h(x) = g(x) + 3 = \tfrac{1}{2}f(x) + 3 = \tfrac{1}{2}\sqrt{x} + 3$. ▪

Note: In Example 2, the order that you perform the transformations is important. If we had translated vertically first, the resulting function would have been $\frac{1}{2}(f(x) + 3)$ or $\frac{1}{2}\sqrt{x} + \frac{3}{2}$.
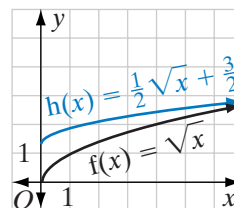
Exercises

Writing About Mathematics

1. Eric said that if $f(x) = |2 - x|$ and $g(x) = |x - 2|$, then $(f + g)(x) = 0$. Do you agree with Eric? Explain why or why not.

2. Give an example of a function g for which $2g(x) \neq g(2x)$. Give an example of a function f for which $2f(x) = f(2x)$.

Developing Skills

3. Let $f = \{(0, 5), (1, 4), (2, 3), (3, 2), (4, 1), (5, 0)\}$
and $g = \{(1, 1), (2, 4), (3, 9), (4, 16), (5, 25)\ (6, 36)\}$.

    a. What is the domain of f?   **b.** What is the domain of g?

    c. What is the domain of $(g - f)$?   **d.** List the ordered pairs of $(g - f)$ in set notation.

    e. What is the domain of $\frac{g}{f}$?   **f.** List the ordered pairs of $\frac{g}{f}$ in set notation.

In 4–7, the graph of $y = f(x)$ is shown. **a.** Find $f(0)$ for each function. **b.** Find x when $f(x) = 0$. For each function, there may be no values, one value, or more than one value. **c.** Sketch the graph of $y = f(x) + 2$. **d.** Sketch the graph of $y = 2f(x)$. **e.** Sketch the graph of $y = -f(x)$.

4.

5.

6.

7.
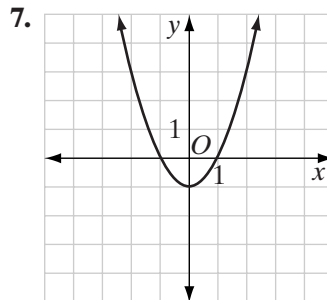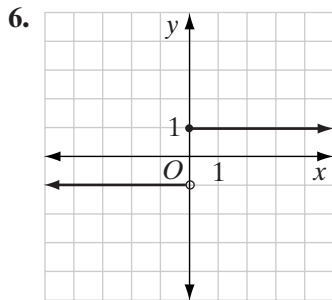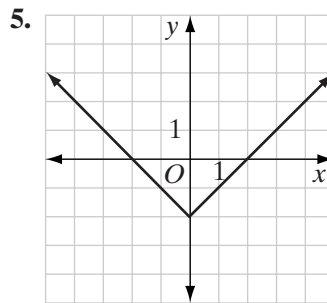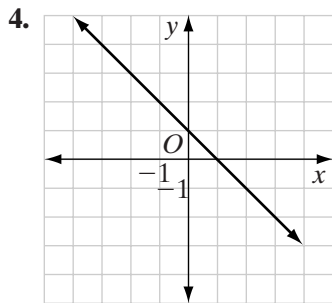
In 8–13, the domain of $f(x) = 4 - 2x$ and of $g(x) = x^2$ is the set of real numbers and the domain of $h(x) = \frac{1}{x}$ is the set of non-zero real numbers.

  a. Write each function value in terms of x.

  b. Find the domain of each function.

    8. $(g + f)(x)$ **9.** $(g - h)(x)$ **10.** $(gh)(x)$

   11. $\left(\frac{g}{f}\right)(x)$ **12.** $(g + 3f)(x)$ **13.** $(-gf)(x)$

Applying Skills

14. When Brian does odd jobs, he charges ten dollars an hour plus two dollars for transportation. When he works for Mr. Atkins, he receives a 15% tip based on his wages for the number of hours that he works but not on the cost of transportation.

   a. Find c(x), the amount Brian charges for x hours of work.

   b. Find t(x), Brian's tip when he works for x hours.

   c. Find e(x), Brian's earnings when he works for Mr. Atkins for x hours, if e(x) = c(x) + t(x).

   d. Find e(3), Brian's earnings when he works for Mr. Atkins for 3 hours.

15. Mrs. Cucci makes candy that she sells for $8.50 a pound. When she ships the candy to out-of-town customers, she adds a flat shipping charge of $2.00 plus an additional $0.50 per pound.

   a. Find c(x), the cost of x pounds of candy.

   b. Find s(x), the cost of shipping x pounds of candy.

   c. Find t(x), the total bill for x pounds of candy that has been shipped, if t(x) = c(x) + s(x).

   d. Find t(5), the total bill for five pounds of candy that is shipped.

16. Create your own function f(x) and show that f(x) + f(x) = 2f(x). Explain why this result is true in general.

4-7 COMPOSITION OF FUNCTIONS

In order to evaluate a function such as $y = \sqrt{x + 3}$, it is necessary to perform two operations: first we must add 3 to x, and then we must take the square root of the sum. The set of ordered pairs $\{(x, y) : y = \sqrt{x + 3}\}$ is the **composition** of two functions. If we let $f(x) = x + 3$ and $g(x) = \sqrt{x}$, we can form the function $h(x) = \sqrt{x + 3}$. We evaluate the function h as shown below:

x	$\xrightarrow{f}$	$x + 3$	$\xrightarrow{g}$	$\sqrt{x + 3}$	$(x, h(x))$
-2	$\xrightarrow{f}$	1	$\xrightarrow{g}$	$\sqrt{1} = 1$	$(-2, 1)$
0	$\xrightarrow{f}$	3	$\xrightarrow{g}$	$\sqrt{3}$	$(0, \sqrt{3})$
1	$\xrightarrow{f}$	4	$\xrightarrow{g}$	$\sqrt{4} = 2$	$(1, 2)$
$\frac{3}{4}$	$\xrightarrow{f}$	$\frac{15}{4}$	$\xrightarrow{g}$	$\sqrt{\frac{15}{4}} = \frac{\sqrt{15}}{2}$	$\left(\frac{3}{4}, \frac{\sqrt{15}}{2}\right)$
5	$\xrightarrow{f}$	8	$\xrightarrow{g}$	$\sqrt{8} = 2\sqrt{2}$	$(5, 2\sqrt{2})$

The function h is a **composite function**. It is the set of ordered pairs that result from mapping x to f(x) and then mapping f(x) to g(f(x)). We say that h(x) = g(f(x)) or that h(x) = g ∘ f(x). The symbol ∘ is read "of." The function h is the composition of g following f. The function f is applied first, followed by g. We evaluate the composition of functions from right to left.

In the introductory example, g is defined only for non-negative real numbers. However, f is a function from the set of real numbers to the set of real numbers. In order for the composition $g \circ f$ to be well defined, we must restrict the domain of f so that its range is a subset of the domain of g. Since the largest possible domain of g is the set of non-negative real numbers, the range of f must be the non-negative real numbers, that is, $f(x) \geq 0$ or $x + 3 \geq 0$ and $x \geq -3$. Therefore, when we restrict f to $\{x : x \geq -3\}$, the composition $g \circ f$ is well defined. The domain of h is $\{x : x \geq -3\}$ and the range of h is $\{y : y \geq 0\}$.

▶ **The domain of the composition $g \circ f$ is the set of all x in the domain of f where f(x) is in the domain of g.**

EXAMPLE I

Given: f $= \{(0, 1), (1, 3), (2, 5), (3, 7)\}$ and g $= \{(1, 0), (3, 3), (5, 1), (7, 2)\}$
Find: **a.** f $\circ$ g   **b.** g $\circ$ f

Solution   **a.** To find $(f \circ g)(x)$, work from right to left. First find g(x) and then find f(g(x)).

x	$\xrightarrow{g}$	g(x)	$\xrightarrow{f}$	$(f \circ g)(x)$
1	$\xrightarrow{g}$	0	$\xrightarrow{f}$	1
3	$\xrightarrow{g}$	3	$\xrightarrow{f}$	7
5	$\xrightarrow{g}$	1	$\xrightarrow{f}$	3
7	$\xrightarrow{g}$	2	$\xrightarrow{f}$	5

b. To find $(g \circ f)(x)$, work from right to left. First find f(x) and then find g(f(x)).

x	$\xrightarrow{f}$	f(x)	$\xrightarrow{g}$	$(g \circ f)(x)$
0	$\xrightarrow{f}$	1	$\xrightarrow{g}$	0
1	$\xrightarrow{f}$	3	$\xrightarrow{g}$	3
2	$\xrightarrow{f}$	5	$\xrightarrow{g}$	1
3	$\xrightarrow{f}$	7	$\xrightarrow{g}$	2

Answers   **a.** f $\circ$ g $= \{(1, 1), (3, 7), (5, 3), (7, 5)\}$
b. g $\circ$ f $= \{(0, 0), (1, 3), (2, 1), (3, 2)\}$

EXAMPLE 2

Let $h(x) = (p \circ q)(x)$, $p(x) = |x|$, and $q(x) = x - 4$.

a. Find h(-3).   **b.** What is the domain of h?   **c.** What is the range of h?

Solution   **a.** $h(-3) = p(q(-3)) = p(-3 - 4) = p(-7) = |-7| = 7$

b. Since the range of q is the same as the domain of p, all elements of the domain of q can be used as the domain of h $= (p \circ q)$. The domain of q is the set of real numbers, so the domain of h is the set of real numbers.

c. All elements of the range of h $= (p \circ q)$ are elements of the range of p. The range of p is the set of non-negative real numbers.

Answers   **a.** $h(-3) = 7$

b. The domain of h is the set of real numbers.

c. The range of h is the set of non-negative real numbers.

EXAMPLE 3

Show that the composition of functions is *not* commutative, that is, it is not necessarily true that $f(g(x)) = g(f(x))$.

Solution If the composition of functions is commutative, then $f(g(x)) = g(f(x))$ for all $f(x)$ and all $g(x)$. To show the composition of functions is *not* commutative, we need to show one counterexample, that is, one pair of functions $f(x)$ and $g(x)$ such that $f(g(x)) \neq g(f(x))$.

Let $f(x) = 2x$ and $g(x) = x + 3$.

$f(g(x)) = f(x + 3)$	$g(f(x)) = g(2x)$
$\qquad = 2(x + 3)$	$\qquad = 2x + 3$
$\qquad = 2x + 6$	
Let $x = 5$.	Let $x = 5$.
Then $f(g(5)) = 2(5) + 6 = 16$.	Then $g(f(5)) = 2(5) + 3 = 13$.

Therefore, $f(g(x)) \neq g(f(x))$ and the composition of functions is not commutative.

EXAMPLE 4

Let $p(x) = x + 5$ and $q(x) = x^2$.

a. Write an algebraic expression for $r(x) = (p \circ q)(x)$.
b. Write an algebraic expression for $f(x) = (q \circ p)(x)$.
c. Evaluate $r(x)$ for $\{x : -2 \leq x \leq 3\}$ if $x \in \{\text{Integers}\}$.
d. Evaluate $f(x)$ for $\{x : -2 \leq x \leq 3\}$ if $x \in \{\text{Integers}\}$.
e. For these two functions, is $(p \circ q)(x) = (q \circ p)(x)$?

Solution **a.** $r(x) = (p \circ q)(x) = p(x^2) = x^2 + 5$
b. $f(x) = (q \circ p)(x) = q(x + 5) = (x + 5)^2 = x^2 + 10x + 25$

c.

x	$r(x)$	$(x, r(x))$
-2	$(-2)^2 + 5 = 9$	$(-2, 9)$
-1	$(-1)^2 + 5 = 6$	$(-1, 6)$
0	$(0)^2 + 5 = 5$	$(0, 5)$
1	$(1)^2 + 5 = 6$	$(1, 6)$
2	$(2)^2 + 5 = 9$	$(2, 9)$
3	$(3)^2 + 5 = 14$	$(3, 14)$

d.

x	$f(x)$	$(x, f(x))$
-2	$(-2)^2 + 10(-2) + 25 = 9$	$(-2, 9)$
-1	$(-1)^2 + 10(-1) + 25 = 16$	$(-1, 16)$
0	$(0)^2 + 10(0) + 25 = 25$	$(0, 25)$
1	$(1)^2 + 10(1) + 25 = 36$	$(1, 36)$
2	$(2)^2 + 10(2) + 25 = 49$	$(2, 49)$
3	$(3)^2 + 10(3) + 25 = 64$	$(3, 64)$

e. No. From the tables, since $(p \circ q)(3) = 14$ and $(q \circ p)(3) = 64$, $(p \circ q)(x) \neq (q \circ p)(x)$.

Answers   **a.** $r(x) = x^2 + 5$ **b.** $f(x) = x^2 + 10x + 25$

c. $\{(-2, 9), (-1, 6), (0, 5), (1, 6), (2, 9), (3, 14)\}$

**d.** $\{(-2, 9), (-1, 16), (0, 25), (1, 36), (2, 49), (3, 64)\}$

e. No

Transformations and Function Composition

Let $p(x) = x + 5$ and $q(x) = x - 5$. The domains of p and q are the set of real numbers and the ranges are also the set of real numbers.

Let $g(x) = x^2$. The domain of g is the set of real numbers and the range is the set of non-negative real numbers.

Let $f(x) = (g \circ q)(x) = g(x + 5) = (x + 5)^2$.

Let $h(x) = (g \circ q)(x) = g(x - 5) = (x - 5)^2$. The domains of f and h are the set of real numbers and the ranges are the set of non-negative real numbers. If we sketch the graph of $g(x) = x^2$ and the graph of $f(x) = (x + 5)^2$, note that $f(x)$ is the graph of $g(x)$ moved 5 units to the left. Similarly, the graph of $h(x) = (x - 5)^2$ is the graph of $g(x)$ moved 5 units to the right.
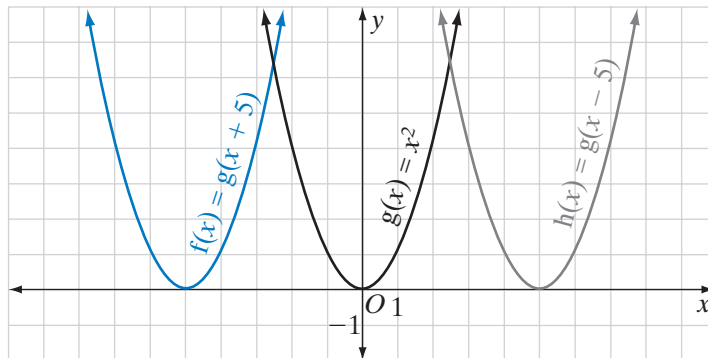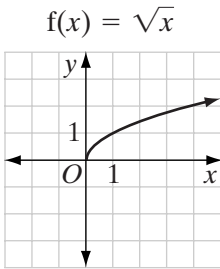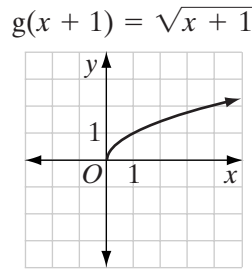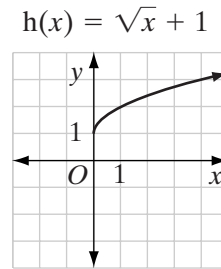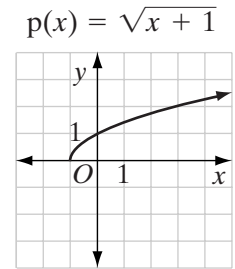

In general, for any function g:

▶ **If $h(x) = g(x + a)$ for a constant a, the graph of $h(x)$ is the graph of $g(x)$ moved $|a|$ units to the left when a is positive and $|a|$ units to the right when a is negative.**

Compare the functions and their graphs on the following page.

$f(x) = \sqrt{x}$

Domain: $x \geq 0$

$g(x + 1) = \sqrt{x + 1}$

Domain: $x \geq -1$

$h(x) = \sqrt{x} + 1$

Domain: $x \geq 0$

$p(x) = \sqrt{x + 1}$

Domain: $x \geq -1$

Note that:

- $g(x)$ is the same function as $f(x)$, $x \to \sqrt{x}$ is the same as $x + 1 \to \sqrt{x + 1}$.
- $h(x)$ is $f(x)$ moved 1 unit up.
- $p(x)$ is $f(x)$ moved 1 unit to the left.

Exercises

Writing About Mathematics

1. Marcie said that if $f(x) = x^2$, then $f(a + 1) = (a + 1)^2$. Do you agree with Marcie? Explain why or why not.

2. Explain the difference between $fg(x)$ and $f(g(x))$.

Developing Skills

In 3–10, evaluate each composition for the given values if $f(x) = 3x$ and $g(x) = x - 2$.

3. $f(g(4))$ **4.** $g(f(4))$ **5.** $f \circ g(-2)$ **6.** $g \circ f(-2)$

7. $f(f(5))$ **8.** $g(g(5))$ **9.** $f\left(g\left(\frac{2}{3}\right)\right)$ **10.** $g\left(f\left(\frac{2}{3}\right)\right)$

In 11–18: **a.** Find $h(x)$ when $h(x) = g(f(x))$. **b.** What is the domain of $h(x)$? **c.** What is the range of $h(x)$? **d.** Graph $h(x)$.

11. $f(x) = 2x + 1, g(x) = 4x$ **12.** $f(x) = 3x, g(x) = x - 1$

13. $f(x) = x^2, g(x) = 4 + x$ **14.** $f(x) = 4 + x, g(x) = x^2$

15. $f(x) = x^2, g(x) = \sqrt{x}$ **16.** $f(x) = |2 + x|, g(x) = -x$

17. $f(x) = 5 - x, g(x) = |x|$ **18.** $f(x) = 3x - 1, g(x) = \frac{1}{3}(x + 1)$

In 19–22, let $f(x) = |x|$. Find $f(g(x))$ and $g(f(x))$ for each given function.

19. $g(x) = x + 3$ **20.** $g(x) = 2x$ **21.** $g(x) = 2x + 3$ **22.** $g(x) = 5 - x$

23. For which of the functions given in 19–22 does $f(g(x)) = g(f(x))$?

24. If $p(x) = 2$ and $q(x) = x + 2$, find $p(q(5))$ and $q(p(5))$.

25. If $h(x) = 2(x + 1)$ and $h(x) = f(g(x))$, what are possible expressions for $f(x)$ and for $g(x)$?

Applying Skills

26. Let $c(x)$ represent the cost of an item, x, plus sales tax and $d(x)$ represent the cost of an item less a discount of $10.

  a. Write $c(x)$ using an 8% sales tax.

  b. Write $d(x)$.

  c. Find $c \circ d(x)$ and $d \circ c(x)$. Does $c \circ d(x) = d \circ c(x)$? If not, explain the difference between the two functions. When does it makes sense to use each function?

  d. Which function can be used to find the amount that must be paid for an item with a $10 discount and 8% tax?

27. The relationship between temperature and the rate at which crickets chirp can be approximated by the function $n(x) = 4x - 160$ where n is the number of chirps per minute and x is the temperature in degrees Fahrenheit. On a given summer day, the temperature outside between the hours of 6 A.M. and 12 P.M. can be modeled by the function $f(x) = 0.55x^2 + 1.66x + 50$ where x is the number of hours elapsed from 6 P.M.

  a. Find the composite function $n \circ f$.

  b. What is the rate of chirping at 11 A.M.?

4-8 INVERSE FUNCTIONS

Identity Function

A function of the form $y = mx + b$ is a linear function. When we let $m = 1$ and $b = 0$, the function is $y = 1x + 0$ or $y = x$. This is a function in which the second element, y, of each ordered pair is equal to the first element.

▶ **The function $y = x$ is called the *identity function*, I, such that $I(x) = x$.**

The graph of the identity function is shown to the right.
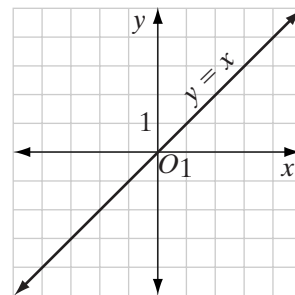
DEFINITION

The **identity function** (**I**) is the function that maps every element of the domain to the same element of the range.

The domain of the identity can be the set of real numbers or any subset of the set of real numbers. The domain and range of the identity function are the same set.

Recall that for the set of real numbers, the identity for addition is 0 because for each real number a, $a + 0 = 0 + a = a$. We can say that adding the identity to any real number a leaves a unchanged.

Similarly, for the set of real numbers, the identity for multiplication is 1 because for each real number a, $a \cdot 1 = 1 \cdot a = a$. We can say that multiplying the identity for multiplication by any real number a leaves a unchanged.

Likewise, the identity function is $I(x)$ because for any function f:

$$(f \circ I) = (I \circ f) = f$$

The composition of the identity function with any function f leaves f unchanged. For example, let $f(x) = 2x + 3$ and $I(x)$ be the identity function, that is, $I(x) = x$.

$(I \circ f)(x)$					$(f \circ I)(x)$			
x $\xrightarrow{f}$	$f(x)$	$\xrightarrow{I}$	$f(x)$		x $\xrightarrow{I}$	x	$\xrightarrow{f}$	$f(x)$
0 $\xrightarrow{f}$	$2(0) + 3 = 3$	$\xrightarrow{I}$	3		0 $\xrightarrow{I}$	0	$\xrightarrow{f}$	$2(0) + 3 = 3$
−1 $\xrightarrow{f}$	$2(-1) + 3 = 1$	$\xrightarrow{I}$	1		−1 $\xrightarrow{I}$	−1	$\xrightarrow{f}$	$2(-1) + 3 = 1$
2 $\xrightarrow{f}$	$2(2) + 3 = 7$	$\xrightarrow{I}$	7		2 $\xrightarrow{I}$	2	$\xrightarrow{f}$	$2(2) + 3 = 7$
5 $\xrightarrow{f}$	$2(5) + 3 = 13$	$\xrightarrow{I}$	13		5 $\xrightarrow{I}$	5	$\xrightarrow{f}$	$2(5) + 3 = 13$
x $\xrightarrow{f}$	$2x + 3$	$\xrightarrow{I}$	$2x + 3$		x $\xrightarrow{I}$	x	$\xrightarrow{f}$	$2x + 3$

$$(I \circ f)(x) = 2x + 3 \text{ and } (f \circ I)(x) = 2x + 3,$$
$$\text{so } (I \circ f)(x) = (f \circ I)(x).$$

Let $f(x)$ be any function whose domain and range are subsets of real numbers. Let $y = x$ or $I(x) = x$ be the identity function. Then:

$$x \xrightarrow{f} f(x) \xrightarrow{I} f(x) \qquad \text{and} \qquad x \xrightarrow{I} x \xrightarrow{f} f(x)$$

Therefore:

$$(I \circ f)(x) = (f \circ I)(x) = f(x)$$

Inverse Functions

In the set of real numbers, two numbers are additive inverses if their sum is the additive identity, 0, and two numbers are multiplicative inverses if their product is the multiplicative identity, 1. We can define inverse functions in a similar way.

The function inverse of f is g if for all x in the domains of $(f \circ g)$ and of $(g \circ f)$, the composition $(f \circ g)(x) = (g \circ f)(x) = I(x) = x$. If g is the function inverse of f, then f is the function inverse of g.

DEFINITION

The functions f and g are **inverse functions** if f and g are functions and $(f \circ g) = (g \circ f) = I$.

For instance, let $f(x) = 2x + 1$ and $g(x) = \frac{x - 1}{2}$.

$(f \circ g)(x)$

x	$\xrightarrow{g}$	$g(x)$	$\xrightarrow{f}$	$f(g(x)) = I(x)$	$(x, I(x))$
-2	$\xrightarrow{g}$	$\frac{-2 - 1}{2} = -\frac{3}{2}$	$\xrightarrow{f}$	$2\left(-\frac{3}{2}\right) + 1 = -3$	$(-3, -3)$
3	$\xrightarrow{g}$	$\frac{3 - 1}{2} = 1$	$\xrightarrow{f}$	$2(1) + 1 = 3$	$(3, 3)$
5	$\xrightarrow{g}$	$\frac{5 - 1}{2} = 2$	$\xrightarrow{f}$	$2(2) + 1 = 5$	$(5, 5)$
x	$\xrightarrow{g}$	$\frac{x - 1}{2}$	$\xrightarrow{f}$	$2\left(\frac{x - 1}{2}\right) + 1 = x$	(x, x)

$(f \circ g)(x) = x$

$(g \circ f)(x)$

| x	$\xrightarrow{f}$	$f(x)$	$\xrightarrow{g}$	$g(f(x)) = I(x)$	$(x, I(x))$
-2	$\xrightarrow{f}$	$2(-2) + 1 = -3$	$\xrightarrow{g}$	$\frac{-3 - 1}{2} = -2$	$(-2. -2)$
3	$\xrightarrow{f}$	$2(3) + 1 = 7$	$\xrightarrow{g}$	$\frac{7 - 1}{2} = 3$	$(3, 3)$
5	$\xrightarrow{f}$	$2(5) + 1 = 11$	$\xrightarrow{g}$	$\frac{11 - 1}{2} = 5$	$(5, 5)$
x	$\xrightarrow{f}$	$2x + 1$	$\xrightarrow{g}$	$\frac{(2x + 1) - 1}{2} = x$	(x, x)

$(g \circ f)(x) = x$

Since $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$, f and g are inverse functions.

The symbol f^{-1} is often used for the inverse of the function f. We showed that when $f(x) = 2x + 1$ and $g(x) = \frac{x - 1}{2}$, f and g are inverse functions. We can write:

$$g(x) = f^{-1}(x) = \frac{x - 1}{2} \qquad \text{or} \qquad f(x) = g^{-1}(x) = 2x + 1$$

The graph of f^{-1} is the reflection of the graph of f over the line $y = x$. For every point (a, b) on the graph of f, there is a point (b, a) on the graph of f^{-1}. For example, $(1, 3)$ is a point on f and $(3, 1)$ is a point on f^{-1}.

If $f = \{(x, y) : y = 2x + 1\}$, then $f^{-1} = \{(x, y) : x = 2y + 1\}$. Since it is customary to write y in terms of x, solve $x = 2y + 1$ for y.
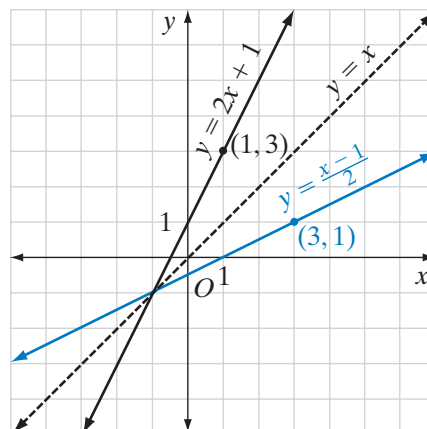
The inverse of a function can be found by interchanging x and y, and then solving for y:

$$y = 2x + 1$$
$$x = 2y + 1$$
$$x - 1 = 2y$$
$$\frac{x - 1}{2} = y$$

In set-builder notation, $f^{-1} = \left\{(x,y):y = \frac{x - 1}{2}\right\}$.

Not every function has an inverse. For instance:

Let $f = \{(-3, 9), (-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4), (3, 9)\}$.

If we interchange the first and second elements of each pair, we obtain the following set:

$$g = \{(9, -3), (4, -2), (1, -1), (0, 0), (1, 1), (4, 2), (9, 3)\}$$

Although g is a relation, it is not a function. Therefore, g cannot be the inverse function of f.

▶ **A function has an inverse function if and only if for every first element there is exactly one second element and for every second element there is exactly one first element. (Alternatively, a function has an inverse if and only it is one-to-one.)**

Since the existence of the inverse of a function depends on whether or not the function is one-to-one, we can use the horizontal line test to determine when a function has an inverse.

Recall that a function that is not one-to-one is said to be many-to-one.

EXAMPLE 1

Let $h(x) = 2 - x$. Find $h^{-1}(x)$ and show that h and h^{-1} are inverse functions.

Solution Let $h = \{(x, y) : y = 2 - x\}$. Then $h^{-1} = \{(x, y) : x = 2 - y\}$.

Solve $x = 2 - y$ for y in terms of x.

$$x = 2 - y$$
$$x - 2 = -y$$
$$-1(x - 2) = -1(-y)$$
$$-x + 2 = y$$
$$h^{-1}(x) = -x + 2$$

Show that $h(h^{-1}(x)) = h^{-1}(h(x)) = I(x)$.

$$h(h^{-1}(x)) = h(-x + 2) \qquad h^{-1}(h(x)) = h^{-1}(2 - x)$$
$$= 2 - (-x + 2) \qquad\qquad = -(2 - x) + 2$$
$$= 2 + x - 2 \qquad\qquad\quad = -2 + x + 2$$
$$= x ✔ \qquad\qquad\qquad\quad = x ✔$$

EXAMPLE 2

If $g(x) = \frac{1}{3}x + 1$, find $g^{-1}(x)$.

Solution   *How to Proceed*

(1) For function g, write y in terms of x: $y = \frac{1}{3}x + 1$

(2) To find the inverse of a function, interchange x and y. Write the function g^{-1} by interchanging x and y: $x = \frac{1}{3}y + 1$

(3) Solve for y in terms of x: $x = \frac{1}{3}y + 1$

$$3x = y + 3$$
$$3x - 3 = y$$

*Answer* $g^{-1}(x) = 3x - 3$

Inverse of an Absolute Value Function

If $y = |x|$ is the absolute value function, then to form an inverse, we interchange x and y. But the relation $x = |y|$ is not a function because for every non-zero value of x, there are two pairs with the same first element. For example, if y is 3, then x is $|3|$ or 3 and one pair of the relation is $(3, 3)$. If y is -3, then x is $|-3|$ or 3 and another pair of the relation is $(3, -3)$. The relation has two pairs, $(3, 3)$ and $(3, -3)$, with the same first element and is not a function. This is shown on the graph at the right.
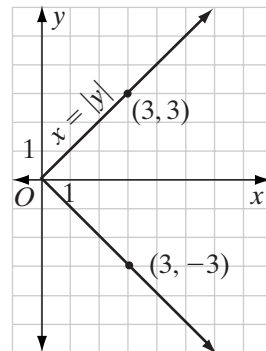
▶ **When the domain of an absolute value function is the set of real numbers, the function is not one-to-one and has no inverse function.**

EXAMPLE 3

The composite function $h = f \circ g$ maps x to $|4 - x|$.

a. What is the function g?

b. What is the function f?

c. Find $h(1)$ and $h(7)$.

d. Is the function h one-to-one?

e. Does h have an inverse function?

Solution **a.** To evaluate this function, we first find $4 - x$. Therefore, $g(x) = 4 - x$.

b. After finding $4 - x$, we find the absolute value. Therefore, $f(x) = |x|$.

c.
$$h(1) = |4 - x| \qquad\qquad h(7) = |4 - x|$$
$$ = |4 - 1| \qquad\qquad = |4 - 7|$$
$$ = |3| \qquad\qquad\qquad = |-3|$$
$$ = 3 \qquad\qquad\qquad = 3$$

d. The function is not one-to-one because at least two pairs, $(1, 3)$ and $(7, 3)$, have the same second element.

e. The function h does not have an inverse because it is not one-to-one. For every ordered pair (x, y) of a function, (y, x) is an ordered pair of the inverse. Two pairs of h are $(1, 3)$ and $(7, 3)$. Two pairs of an inverse would be $(3, 1)$ and $(3, 7)$. A function cannot have two pairs with the same first element. Therefore, h has no inverse function. Any function that is not one-to-one cannot have an inverse function.

Inverse of a Quadratic Function

The domain of a polynomial function of degree two (or a quadratic function) is the set of real numbers. The range is a subset of the real numbers. A polynomial function of degree two is *not* one-to-one and is *not* onto.

A function that is not one-to-one does not have an inverse function under composition. However, it is sometimes possible to select a subset of the domain for which the function is one-to-one and does have an inverse for that subset of the original domain. For example, we can restrict the domain of $y = x^2 + 2x - 3$ to those values of x that are greater than or equal to the x coordinate of the turning point.

For $x \geq -1$, there is exactly one y value for each x value. Some of the pairs of the function $y = x^2 + 2x - 3$ when $x \geq -1$ are $(-1, -4)$, $(0, -3)$, $(1, 0)$, and $(2, 5)$. Then $(-4, -1)$, $(-3, 0)$, $(0, 1)$, and $(5, 2)$ would be pairs of the inverse function, $x = y^2 + 2y - 3$, when $y \geq -1$.

Note that the inverse is the reflection of the function over the line $y = x$.
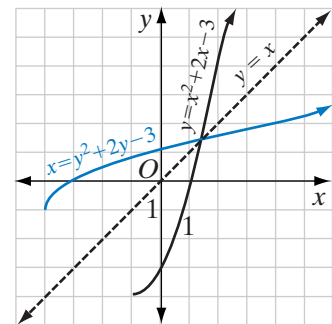
Exercises

Writing About Mathematics

1. Taylor said that if (a, b) is a pair of a one-to-one function f, then (b, a) must be a pair of the inverse function f^{-1}. Do you agree with Taylor? Explain why or why not.

2. Christopher said that $f(x) = |x - 2|$ and $g(x) = |x + 2|$ are inverse functions after he showed that $f(g(2)) = 2$, $f(g(5)) = 5$, and $f(g(7)) = 7$. Do you agree that f and g are inverse functions? Explain why or why not.

Developing Skills

In 3–10, find each of the function values when $f(x) = 4x$.

3. I(3) **4.** I(5) **5.** I(−2) **6.** I(f(2))

7. f(I(3)) **8.** f(f^{-1}(−6)) **9.** f^{-1}(f(−6)) **10.** f$\left(f^{-1}(\sqrt{2})\right)$

In 11–16, determine if the function has an inverse. If so, list the pairs of the inverse function. If not, explain why there is no inverse function.

11. $\{(0, 8), (1, 7), (2, 6), (3, 5), (4, 4)\}$ **12.** $\{(1, 4), (2, 7), (1, 10), (4, 13)\}$

13. $\{(0, 8), (2, 6), (4, 4), (6, 2)\}$ **14.** $\{(2, 7), (3, 7), (4, 7), (5, 7), (6, 7)\}$

15. $\{(-1, 3), (-1, 5), (-2, 7), (-3, 9), (-4, 11)\}$ **16.** $\{(x, y) : y = x^2 + 2 \text{ for } 0 \leq x \leq 5\}$

In 17–20: **a.** Find the inverse of each given function. **b.** Describe the domain and range of each given function and its inverse in terms of the largest possible subset of the real numbers.

17. $f(x) = 4x - 3$ **18.** $g(x) = x - 5$ **19.** $f(x) = \frac{x + 5}{3}$ **20.** $f(x) = \sqrt{x}$

21. If $f = \{(x, y) : y = 5x\}$ is a direct variation function, find f^{-1}.

22. If $g = \{(x, y) : y = 7 - x\}$, find g^{-1} if it exists. Is it possible for a function to be its own inverse?

23. Does $y = x^2$ have an inverse function if the domain is the set of real numbers? Justify your answer.

In 24–26, sketch the inverse of the given function.

24.

25.

26.
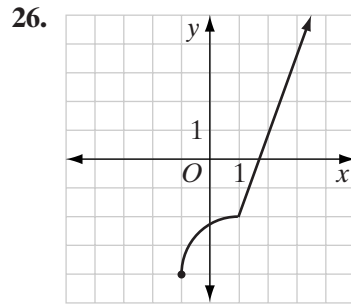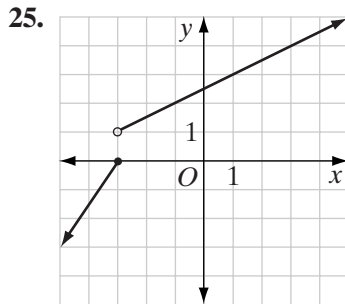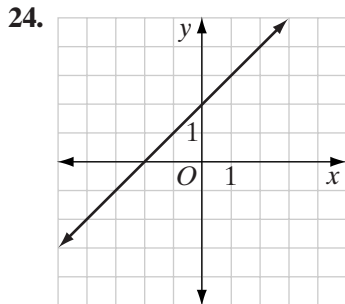
Applying Skills

27. On a particular day, the function that converts American dollars, x, to Indian rupees, $f(x)$, is $f(x) = 0.2532x$. Find the inverse function that converts rupees to dollars. Verify that the functions are inverses.

28. When the function $g(x) = x^2 + 8x + 18$ is restricted to the interval $x \geq 2$, the inverse is $g^{-1}(x) = \sqrt{x - 2} - 4$.

a. Graph g for values of $x \geq 2$. Graph g^{-1} on the same set of axes.

b. What is the domain of g? What is the range of g?

c. What is the domain of g^{-1}? What is the range of g^{-1}?

d. Describe the relationship between the domain and range of g and its inverse.

4-9 CIRCLES

A **circle** is the set of points at a fixed distance from a fixed point. The fixed distance is the **radius** of the circle and the fixed point is the **center** of the circle.

The length of the line segment through the center of the circle with endpoints on the circle is a **diameter** of the circle. The definition of a circle can be used to write an equation of a circle in the coordinate plane.

Center-Radius Form of the Equation of a Circle

Recall the formula for the distance between two points in the coordinate plane. Let $B(x_1, y_1)$ and $A(x_2, y_2)$ be any two points in the coordinate plane. $C(x_2, y_1)$ is the point on the same vertical line as A and on the same horizontal line as B. The length of the line segment joining two points on the same horizontal line is the absolute value of the difference of their x-coordinates. Therefore, $BC = |x_1 - x_2|$.
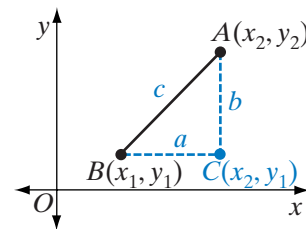
The length of the line segment joining two points on the same vertical line is the absolute value of the difference of their y-coordinates. Therefore, $AC = |y_1 - y_2|$.

Triangle ABC is a right triangle whose hypotenuse is $\overline{AB}$. Use the Pythagorean Theorem with $a = BC$, $b = AC$, and $c = AB$ to express AB in terms of the coordinates of its endpoints.

$$c^2 = a^2 + b^2$$
$$(AB)^2 = (BC)^2 + (AC)^2$$
$$(AB)^2 = (|x_1 - x_2|)^2 + (|y_1 - y_2|)^2$$
$$AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

If d is the distance from A to B, then:
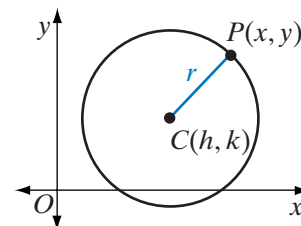
$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

To write the equation for the circle of radius r whose center is at $C(h, k)$, let $P(x, y)$ be any point on the circle. Using the distance formula, CP is equal to

$$CP = \sqrt{(x - h)^2 + (y - k)^2}$$

However, CP is also equal to the radius, r.

$$\sqrt{(x - h)^2 + (y - k)^2} = r$$
$$\left(\sqrt{(x - h)^2 + (y - k)^2}\right)^2 = r^2$$

$$(x - h)^2 + (y - k)^2 = r^2$$

This is the **center-radius form of the equation of a circle**.

For any circle with radius r and center at the origin, $(0, 0)$, we can write the equation of the circle by letting $h = 0$ and $k = 0$.

$$(x - h)^2 + (y - k)^2 = r^2$$
$$(x - 0)^2 + (y - 0)^2 = r^2$$

$$x^2 + y^2 = r^2$$

The equation of the circle with radius 3 and center at the origin is
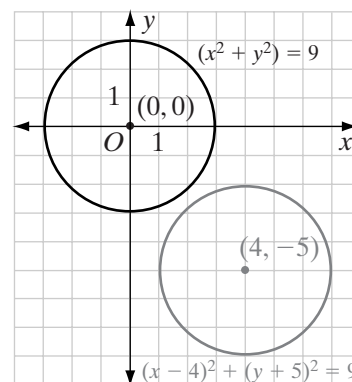
$$(x - 0)^2 + (y - 0)^2 = (3)^2$$

or

$$x^2 + y^2 = 9$$

The graph intersects the x-axis at $(-3, 0)$ and $(3, 0)$ and the y-axis at $(0, -3)$ and $(0, 3)$. The equation of the circle with radius 3 and center at $(4, -5)$ is

$$(x - 4)^2 + (y - (-5))^2 = 3^2$$

or

$$(x - 4)^2 + (y + 5)^2 = 9$$

Note that when the graph of the circle with radius 3 and center at the origin is moved 4 units to the right and 5 units down, it is the graph of the equation $(x - 4)^2 + (y + 5)^2 = 9$.

A circle is drawn on the coordinate plane. We can write an equation of the circle if we know the coordinates of the center and the coordinates of one point on the circle. In the diagram, the center of the circle is at $C(-2, 3)$ and one point on the circle is $P(1, 4)$. To find the equation of the circle, we can use the distance formula to find CP, the radius of the circle.
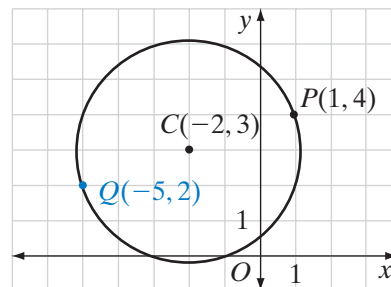
$$CP = \sqrt{(-2 - 1)^2 + (3 - 4)^2}$$
$$= \sqrt{(-3)^2 + (-1)^2}$$
$$= \sqrt{9 + 1}$$
$$= \sqrt{10}$$

Therefore, the equation of the circle is

$$(x - (-2))^2 + (y - 3)^2 = (\sqrt{10})^2$$
$$(x + 2)^2 + (y - 3)^2 = 10$$

Note that $Q(-5, 2)$ is also a point on the circle. If we substitute the coordinates of Q for x and y in the equation, we will have a true statement.

$$(x + 2)^2 + (y - 3)^2 = 10$$
$$(-5 + 2)^2 + (2 - 3)^2 \stackrel{?}{=} 10$$
$$(-3)^2 + (-1)^2 \stackrel{?}{=} 10$$
$$9 + 1 \stackrel{?}{=} 10$$
$$10 = 10 ✔$$

Standard Form of the Equation of a Circle

The **standard form of the equation of a circle** is:

$$x^2 + y^2 + Dx + Ey + F = 0$$

We can write the center-radius form in standard form by expanding the squares of the binomials.

$$(x - h)^2 + (y - k)^2 = r^2$$
$$x^2 - 2hx + h^2 + y^2 - 2ky + k^2 = r^2$$

This equation in standard form is $x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0$.

Note that in the standard form of the equation of a circle, the coefficients of x^2 and y^2 are both equal and usually equal to 1. If we compare the two equations, $D = -2h$, $E = -2k$, and $F = h^2 + k^2 - r^2$.

EXAMPLE 1

a. Find the center-radius form of the equation $x^2 + y^2 - 4x + 6y - 3 = 0$.

b. Determine the center and radius of the circle.

Solution **a.** $D = -4 = -2h$. Therefore, $h = 2$ and $h^2 = 4$.

$E = 6 = -2k$. Therefore, $k = -3$ and $k^2 = 9$.

$F = -3 = h^2 + k^2 - r^2$. Therefore, $-3 = 4 + 9 - r^2$ or $r^2 = 16$.

The equation of the circle in center-radius form is $(x - 2)^2 + (y + 3)^2 = 16$.

b. This is a circle with center at $(2, -3)$ and radius equal to $\sqrt{16} = 4$.

Answers **a.** $(x - 2)^2 + (y + 3)^2 = 16$ **b.** Center $= (2, -3)$, radius $= 4$

EXAMPLE 2

The endpoints of a diameter of a circle are $P(6, 1)$ and $Q(-4, -5)$. Write the equation of the circle: **a.** in center-radius form **b.** in standard form.

Solution **a.** The center of a circle is the midpoint of a diameter. Use the midpoint formula to find the coordinates of the center, M.

$$M(x_m, y_m) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$
$$= \left(\frac{6 + (-4)}{2}, \frac{1 + (-5)}{2}\right)$$
$$= \left(\frac{2}{2}, \frac{-4}{2}\right)$$
$$= (1, -2)$$

Use the distance formula to find MP, the radius, r, of the circle.

Let $(x_1, y_1) = (6, 1)$, the coordinates of P and $(x_2, y_2) = (1, -2)$, the coordinates of M.

$$r = MP = \sqrt{(6 - 1)^2 + (1 - (-2))^2}$$
$$= \sqrt{5^2 + 3^2}$$
$$= \sqrt{25 + 9}$$
$$= \sqrt{34}$$

Substitute the coordinates of the center and the radius of the circle in the center-radius form of the equation.

$$(x - h)^2 + (y - k)^2 = r^2$$
$$(x - 1)^2 + (y - (-2))^2 = (\sqrt{34})^2$$
$$(x - 1)^2 + (y + 2)^2 = 34$$

b. Expand the square of the binomials of the center-radius form of the equation.

$$(x - 1)^2 + (y + 2)^2 = 34$$
$$x^2 - 2x + 1 + y^2 + 4y + 4 = 34$$
$$x^2 + y^2 - 2x + 4y + 1 + 4 - 34 = 0$$
$$x^2 + y^2 - 2x + 4y - 29 = 0$$

Answers **a.** $(x - 1)^2 + (y + 2)^2 = 34$ **b.** $x^2 + y^2 - 2x + 4y - 29 = 0$

EXAMPLE 3

In standard form, the equation of a circle is $x^2 + y^2 + 3x - 4y - 14 = 0$. Write the equation in center-radius form and find the coordinates of the center and radius of the circle.

Solution *How to Proceed*

(1) Complete the square of $x^2 + 3x$. We "complete the square" by writing the expression as a trinomial that is a perfect square. Since $(x + a)^2 = x^2 + 2ax + a^2$, the constant term needed to complete the square is the square of one-half the coefficient of x:

$$x^2 + 3x + \left(\tfrac{3}{2}\right)^2 = x^2 + 3x + \tfrac{9}{4}$$
$$= \left(x + \tfrac{3}{2}\right)^2$$

(2) Complete the square of $y^2 - 4y$. The constant term needed to complete the square is the square of one-half the coefficient of y:

$$y^2 - 4y + \left(\tfrac{-4}{2}\right)^2 = y^2 - 4y + 4$$
$$= (y - 2)^2$$

(3) Add 14 to both sides of the given equation:

$$x^2 + y^2 + 3x - 4y - 14 = 0$$
$$x^2 + y^2 + 3x - 4y = 14$$

(4) To both sides of the equation, add the constants needed to complete the squares:

$$x^2 + 3x + \tfrac{9}{4} + y^2 - 4y + 4 = 14 + \tfrac{9}{4} + 4$$

(5) Write the equation in center-radius form. Determine the center and radius of the circle from the center-radius form:

$$\left(x + \tfrac{3}{2}\right)^2 + (y - 2)^2 = \tfrac{56}{4} + \tfrac{9}{4} + \tfrac{16}{4}$$
$$\left(x + \tfrac{3}{2}\right)^2 + (y - 2)^2 = \tfrac{81}{4}$$
$$\left(x + \tfrac{3}{2}\right)^2 + (y - 2)^2 = \left(\tfrac{9}{2}\right)^2$$

Answer The equation of the circle in center-radius form is $\left(x + \tfrac{3}{2}\right)^2 + (y - 2)^2 = \left(\tfrac{9}{2}\right)^2$.

The center is at $\left(-\tfrac{3}{2}, 2\right)$ and the radius is $\tfrac{9}{2}$.

Exercises

Writing About Mathematics

1. Is the set of points on a circle a function? Explain why or why not.

2. Explain why $(x - h)^2 + (y - k)^2 = -4$ is not the equation of a circle.

Developing Skills

In 3–10, the coordinates of point P on the circle with center at C are given. Write an equation of each circle: **a.** in center-radius form **b.** in standard form.

3. $P(0, 2), C(0, 0)$ **4.** $P(0, -3), C(0, 0)$ **5.** $P(4, 0), C(0, 0)$ **6.** $P(3, 2), C(4, 2)$

7. $P(-1, 5), C(-1, 1)$ **8.** $P(0, -3), C(6, 5)$ **9.** $P(1, 1), C(6, 13)$ **10.** $P(4, 2), C(0, 1)$

In 11–19, write the equation of each circle.

11.

12.

13.

14.

15.

16.

17.

18.

19.
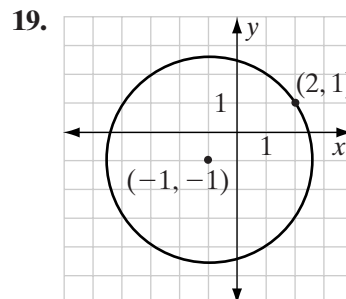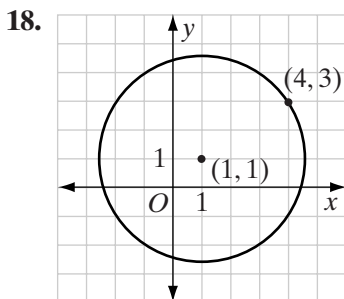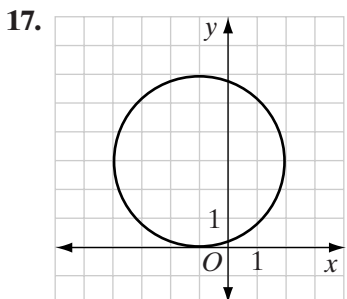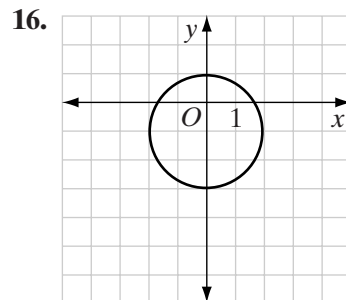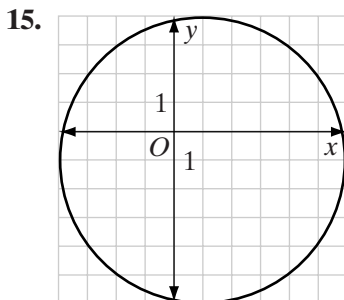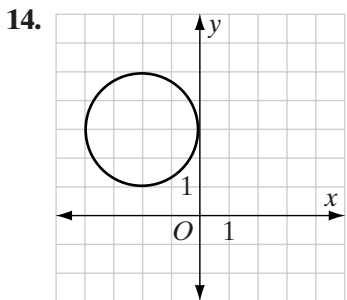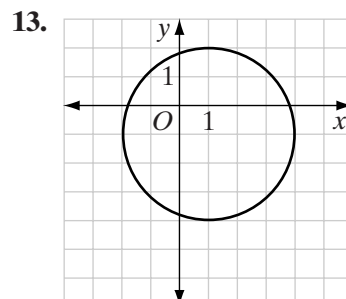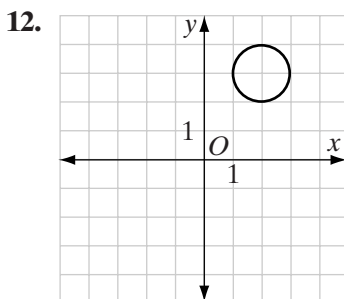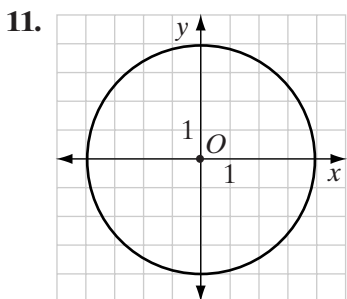
In 20–27: **a.** Write each equation in center-radius form. **b.** Find the coordinates of the center. **c.** Find the radius of the circle.

20. $x^2 + y^2 - 25 = 0$

21. $x^2 + y^2 - 2x - 2y - 7 = 0$

22. $x^2 + y^2 + 2x - 4y + 1 = 0$

23. $x^2 + y^2 - 6x + 2y - 6 = 0$

24. $x^2 + y^2 + 6x - 6y + 6 = 0$

25. $x^2 + y^2 - 8y = 0$

26. $x^2 + y^2 + 10x - 5y - 32 = 0$

27. $x^2 + y^2 + x - 3y - 2 = 0$

Applying Skills

28. An architect is planning the entryway into a courtyard as an arch in the shape of a semi-circle with a radius of 8 feet. The equation of the arch can be written as $(x - 8)^2 + y^2 = 64$ when the domain is the set of non-negative real numbers less than or equal to 16 and the range is the set of positive real numbers less than or equal to 8.

   a. Draw the arch on graph paper.

   b. Can a box in the shape of a cube whose edges measure 6 feet be moved through the arch?

   c. Can a box that is a rectangular solid that measures 8 feet by 8 feet by 6 feet be moved through the arch?

29. What is the measure of a side of a square that can be drawn with its vertices on a circle of radius 10?

30. What are dimensions of a rectangle whose length is twice the width if the vertices are on a circle of radius 10?

31. Airplane passengers have been surprised to look down over farmland and see designs, called *crop circles*, cut into cornfields. Suppose a farmer wishes to make a design consisting of three **concentric circles**, that is, circles with the same center but different radii. Write the equations of three concentric circles centered at a point in a cornfield with coordinates $(2, 2)$.

Hands-On-Activity

Three points determine a circle. If we know the coordinates of three points, we should be able to determine the equation of the circle through these three points. The three points are the vertices of a triangle. Recall that the perpendicular bisectors of the sides of a triangle meet in a point that is equidistant from the vertices of the triangle. This point is the center of the circle through the three points.

   To find the circle through the points $P(5, 3)$, $Q(-3, 3)$, and $R(3, -3)$, follow these steps. Think of P, Q, and R as the vertices of a triangle.

   1. Find the midpoint of $\overline{PQ}$.

   2. Find the slope of $\overline{PQ}$ and the slope of the line perpendicular to $\overline{PQ}$.

   3. Find the equation of the perpendicular bisector of $\overline{PQ}$.

   4. Repeat steps 1 through 3 for $\overline{QR}$.

   5. Find the point of intersection of the perpendicular bisectors of $\overline{PQ}$ and $\overline{QR}$. Call this point C, the center of the circle.

   6. Find $CP = CQ = CR$. The distance from C to each of these points is the radius of the circle.

   7. Use the coordinates of C and the radius of the circle to write the equation of the circle. Show that P, Q, and R are points on the circle by showing that their coordinates make the equation of the circle true.

4-10 INVERSE VARIATION

Forty people live in an apartment complex. They often want to divide them-selves into groups to gather information that will be of interest to all of the ten-ants. In the past they have worked in 1 group of 40, 2 groups of 20, 4 groups of 10, 5 groups of 8, 8 groups of 5, and 10 groups of 4. Notice that as the number of groups increases, the number of people in each group decreases and when the number of groups decreases, the number of people in each group increases. We say that the number of people in each group and the number of groups are *inversely proportional* or that the two quantities *vary inversely*.

DEFINITION

Two numbers, x and y, **vary inversely** or are **inversely proportional** when $xy = c$, a non-zero constant.

Let x_1 and y_1 be two numbers such that $x_1 y_1 = c$, a constant. Let x_2 and y_2 be two other numbers such that $x_2 y_2 = c$, the same constant. Then:

$$x_1 y_1 = x_2 y_2 \qquad \text{or} \qquad \frac{x_1}{x_2} = \frac{y_2}{y_1}$$

Note that when this **inverse variation** relationship is written as a proportion, the order of the terms in x is opposite to the order of the terms in y.

If x and y vary inversely and $xy = 20$, then the domain and range of this rela-tion are the set of non-zero real numbers. The graph of this relation is shown below.

x	y
20	1
10	2
5	4
4	5
2	10
1	20

x	y
−20	−1
−10	−2
−5	−4
−4	−5
−2	−10
−1	−20
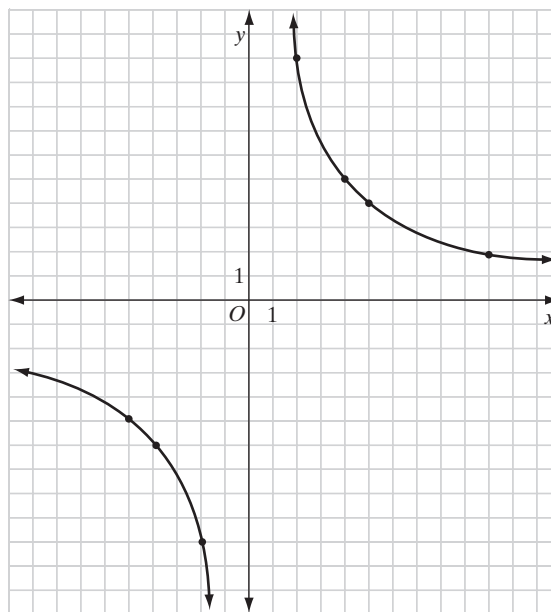
Note:
1. No vertical line intersects the graph in more than one point, so the relation is a function.

2. No horizontal line intersects the graph in more than one point, so the function is one-to-one.

3. This is a function from the set of non-zero real numbers to the set of non-zero real numbers. The range is the set of non-zero real numbers. The function is onto.

4. If we interchange x and y, the function remains unchanged. The function is its own inverse.

5. The graph is called a **hyperbola**.

EXAMPLE 1

The areas of $\triangle ABC$ and $\triangle DEF$ are equal. The length of the base of $\triangle ABC$ is four times the length of the base of $\triangle DEF$. How does the length of the altitude to the base of $\triangle ABC$ compare with the length of the altitude to the base of $\triangle DEF$?

Solution Let b_1 be the length of the base and h_1 be the height of $\triangle ABC$.

Let b_2 be the length of the base and h_2 be the height of $\triangle DEF$.

(1) Area of $\triangle ABC$ = area of $\triangle DEF$. $\qquad \frac{1}{2}b_1h_1 = \frac{1}{2}b_2h_2$

The lengths of the bases and the $\qquad\qquad b_1h_1 = b_2h_2$
heights of these triangles are
inversely proportional:

(2) The length of the base of $\triangle ABC$ $\qquad\qquad b_1 = 4b_2$
is four times the length of the base
of $\triangle DEF$:

(3) Substitute $4b_2$ for b_1 in the equation $\qquad 4b_2h_1 = b_2h_2$
$b_1h_1 = b_2h_2$:

(4) Divide both sides of the equation $\qquad\qquad h_1 = \frac{1}{4}h_2$
by $4b_2$:

Answer The height of $\triangle ABC$ is one-fourth the height of $\triangle DEF$.

Note: When two quantities are inversely proportional, as one quantity changes by a factor of a, the other quantity changes by a factor of $\frac{1}{a}$.

EXAMPLE 2

Mrs. Vroman spends \$3.00 to buy cookies for her family at local bakery. When the price of a cookie increased by 10 cents, the number of cookies she bought decreased by 1. What was the original price of a cookie and how many did she buy? What was the increased price of a cookie and how many did she buy?

Solution   For a fixed total cost, the number of items purchased varies inversely as the cost of one item.

Let $x_1 = p =$ the original price of one cookie in cents

$y_1 = n =$ the number of cookies purchased at the original price

$x_2 = p + 10 =$ the increased price of one cookie in cents

$y_2 = n - 1 =$ the number of cookies purchased at the increased price

(1) Use the product form for inverse variation:
$$x_1 y_1 = x_2 y_2$$

(2) Substitute values in terms of x and y:
$$pn = (p + 10)(n - 1)$$

(3) Multiply the binomials:
$$pn = pn - p + 10n - 10$$

(4) Solve for p in terms of n:
$$0 = -p + 10n - 10$$
$$p = 10n - 10$$

(5) Since she spends \$3 or 300 cents for cookies, $pn = 300$. Write this equation in terms of n by substituting $10n - 10$ for p:
$$pn = 300$$
$$(10n - 10)(n) = 300$$
$$10n^2 - 10n = 300$$
$$10n^2 - 10n - 300 = 0$$

(6) Solve for n:
$$n^2 - n - 30 = 0$$
$$(n - 6)(n + 5) = 0$$
$$n - 6 = 0 \mid n + 5 = 0$$
$$n = 6 \mid \quad n = -5 \text{ (reject this root)}$$

(7) Solve for p:
$$p = 10n - 10$$
$$p = 10(6) - 10$$
$$p = 50$$

Answer   Originally a cookie cost 50 cents and Mrs. Vroman bought 6 cookies. The price increased to 60 cents and she bought 5 cookies.
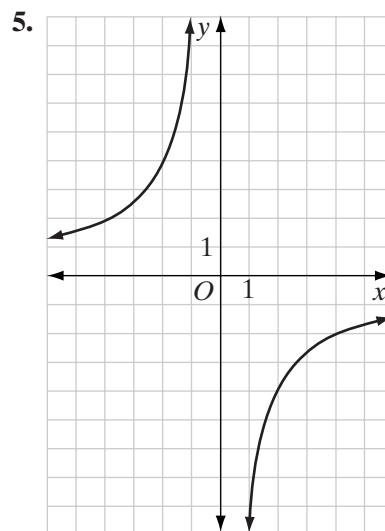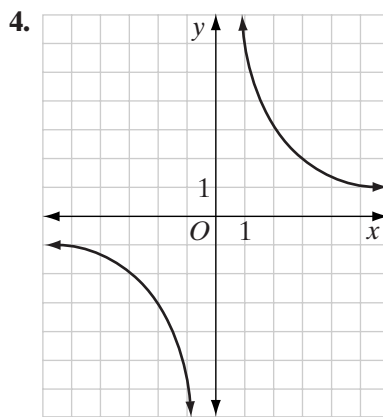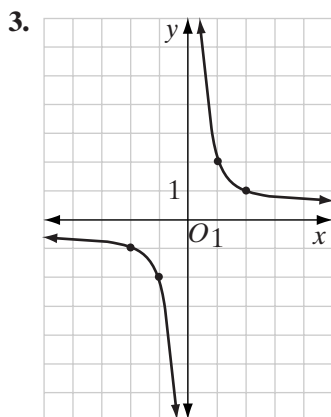
Exercises

Writing About Mathematics

1. Is the function $f = \{(x, y) : xy = 20\}$ a polynomial function? Explain why or why not.

2. Explain the difference between direct variation and inverse variation.

Developing Skills

In 3–5, each graph is a set of points whose x-coordinates and y-coordinates vary inversely. Write an equation for each function.

3.

4.

5.

In 6–12, tell whether the variables vary directly, inversely, or neither.

6. On his way to work, Randy travels 10 miles at r miles per hour for h hours.

7. A driver travels for 4 hours between stops covering d miles at a rate of r miles per hour.

8. Each day, Jaymee works for 6 hours typing p pages of a report at a rate of m minutes per page.

9. Each day, Sophia works for h hours typing p pages of a report at a rate of 15 minutes per page.

10. Each day, Brandon works for h hours typing 40 pages of a report at a rate of m minutes per page.

11. Each day, I am awake a hours and I sleep s hours.

12. A bank pays 4% interest on all savings accounts. A depositor receives I dollars in interest when the balance in the savings account is P dollars.

Applying Skills

13. Rectangles *ABCD* and *EFGH* have the same area. The length of *ABCD* is equal to twice the length of the *EFGH*. How does the width of *ABCD* compare to the width of *EFGH*?

14. Aaron can ride his bicycle to school at an average rate that is three times that of the rate at which he can walk to school. How does the time that it takes Aaron to ride to school compare with the time that it takes him to walk to school?

15. When on vacation, the Ross family always travels the same number of miles each day.

a. Does the time that they travel each day vary inversely as the rate at which they travel?

b. On the first day the Ross family travels for 3 hours at an average rate of 60 miles per hour and on the second day they travel for 4 hours. What was their average rate of speed on the second day?

16. Megan traveled 165 miles to visit friends. On the return trip she was delayed by construction and had to reduce her average speed by 22 miles per hour. The return trip took 2 hours longer. What was the time and average speed for each part of the trip?

17. Ian often buys in large quantities. A few months ago he bought several cans of frozen orange juice for $24. The next time Ian purchased frozen orange juice, the price had increased by $0.10 per can and he bought 1 less can for the same total price. What was the price per can and the numbers of cans purchased each time?

CHAPTER SUMMARY

A **relation** is a set of ordered pairs.

A **function** is a relation such that no two ordered pairs have the same first element. A **function from set *A* to set *B*** is the set of ordered pairs (x, y) such that every $x \in A$ is paired with exactly one $y \in B$. Set *A* is the **domain** of the function. The **range** of the function is a subset of set *B* whose elements are the second elements of the pairs of the function. A function is **onto** if the range is set *B*.

Function Notation

1. $f = \{(x, y) : y = 2x + 3\}$ The function f is the set of ordered pairs $(x, 2x + 3)$.

2. $f(x) = 2x + 3$ The function value paired with x is $2x + 3$.

3. $y = 2x + 3$ The second element of the pair (x, y) is $2x + 3$.

4. $\{(x, 2x + 3)\}$ The set of ordered pairs whose second element is 3 more than twice the first.

5. $f: x \rightarrow 2x + 3$ Under the function f, x maps to $2x + 3$.

6. $x \overset{f}{\rightarrow} 2x + 3$ The image of x for the function f is $2x + 3$.

In general, if f(x) and g(x) are functions,

$$(f + g)(x) = f(x) + g(x)$$

$$(f - g)(x) = f(x) - g(x)$$

$$(fg)(x) = f(x)g(x)$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

The domain of $(f + g), (f - g)$, and (fg) is the set of elements common to the domain of f and of g, and the domain of $\frac{f}{g}$ is the set of elements common to the domains of f and of g for which $g(x) \neq 0$.

A **composite function** $h(x) = g(f(x))$ is the set of ordered pairs that are the result of mapping x to $f(x)$ and then mapping $f(x)$ to $g(f(x))$. The function $h(x) = g(f(x))$ can also be written as $g \circ f(x)$. The function h is the **composition** of g following f.

The **identity function, I(x)**, is that function that maps every element of the domain to the same element of the range.

A function f is said to be **one-to-one** if no two ordered pairs have the same second element.

Every one-to-one function $f(x)$ has an inverse function $f^{-1}(x)$ if $f(f^{-1}(x)) = f^{-1}(f(x)) = I(x)$. The range of f is the domain of f^{-1} and the range of f^{-1} is the domain of f.

When the ratio of two variables is a constant, we say that the variables are **directly proportional** or that the variables **vary directly**.

When the domain of an absolute value function, $y = |x|$, is the set of real numbers, the function is not one-to-one and has no inverse function.

A **polynomial function of degree n** is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0, a_n \neq 0$$

The **roots** or **zeros** of a polynomial function are the values of x for which $y = 0$. These are the x-coordinates of the points at which the graph of the function intersects the x-axis.

The **center-radius form** of the equation of a circle is $(x - h)^2 + (y - k)^2 = r^2$ where (h, k) is the center of the circle and r is the radius. The **standard form** of the equation of a circle is:

$$x^2 + y^2 + Dx + Ey + F = 0$$

Two numbers x and y **vary inversely** or are **inversely proportional** when $xy = a$ and a is a non-zero constant. The domain and range of $xy = a$ or $y = \frac{a}{x}$ are the set of non-zero real numbers.

For every real number a, the graph of $f(x + a)$ is the graph of $f(x)$ moved $|a|$ units to the left when a is positive and $|a|$ units to the right when a is negative.

For every real number a, the graph of $f(x) + a$ is the graph of $f(x)$ moved $|a|$ units up when a is positive and $|a|$ units down when a is negative.

The graph of $-f(x)$ is the graph of $f(x)$ reflected in the x-axis.

For every positive real number a, the graph of $af(x)$ is the graph of $f(x)$ stretched by a factor of a in the y direction if $a > 1$ or compressed by a factor of a if $0 < a < 1$.

VOCABULARY

4-1 Set-builder notation • Relation • Function • Domain of a function • Range of a function • Function from set A to set B • $\in$ • Onto • Vertical line test

4-3 Linear Function • Horizontal line test • One-to-one • Directly proportional • Vary directly • Direct variation

4-4 Absolute value function • Many-to-one

4-5 Polynomial function f of degree n • Quadratic function • Parabola • Turning point • Vertex • Axis of symmetry of the parabola • Roots • Zeros • Double root

4-7 Composition • Composite function

4-8 Identity function (I) • Inverse functions • f^{-1}

4-9 Circle • Radius • Center • Diameter • Center-radius form of the equation of a circle • Standard form of the equation of a circle • Concentric circles

4-10 Vary inversely • Inversely proportional • Inverse variation • Hyperbola

REVIEW EXERCISES

In 1–5: Determine if the relation is or is not a function. Justify your answer.

1. $\{(0, 0), (1, 1), (1, -1), (4, 2), (4, -2), (3, 4), (5, 7)\}$

2. $\{(x, y) : y = \sqrt{x}\,\}$ **3.** $\{(x, y) : x^2 + y^2 = 9\}$

4. $\{(x, y) : y = 2x + 3\}$ **5.** $\{(x, y) : y = -\frac{4}{x}\}$

In 6–11, each equation defines a function from the set of real numbers to the set of real numbers. **a.** Is the function one-to-one? **b.** Is the function onto? **c.** Does the function have an inverse function? If so, write an equation of the inverse.

6. $y = 4x + 3$ **7.** $y = x^2 - 2x + 5$

8. $y = |x + 2|$ **9.** $y = x^3$

10. $y = -1$ **11.** $y = \sqrt{x - 4}$

In 12–17, from each graph, determine any rational roots of the function.

12.

13.

14.

15.

16.

17.
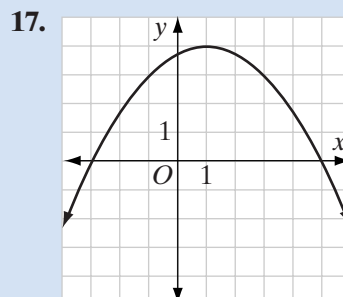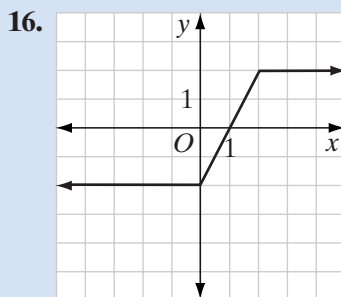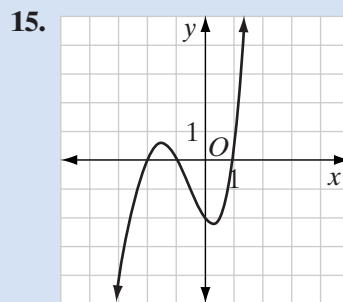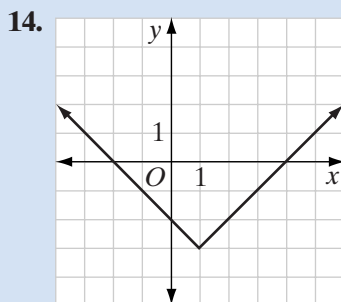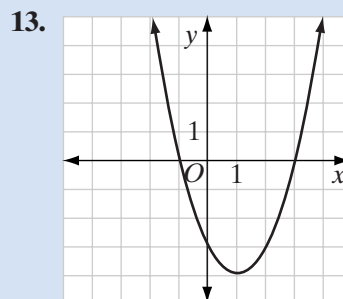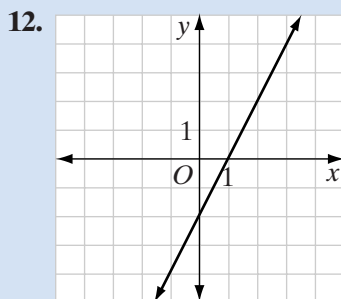
18. What are zeros of the function $f(x) = |x + 3| - 4$?

In 19–22, $p = \{(0, 6), (1, 5), (2, 4), (3, 3), (4, 2)\}$ and $q = \{(0, 1), (1, 3), (2, 5), (3, 7), (4, 9)\}$. Find:

19. $p + q$　　　　　**20.** $p - q$　　　　　**21.** pq　　　　　**22.** $\frac{p}{q}$

In 23–30, for each function f and g, find $f(g(x))$ and $g(f(x))$.

23. $f(x) = 2x$ and $g(x) = x + 3$　　　　**24.** $f(x) = |x|$ and $g(x) = 4x - 1$

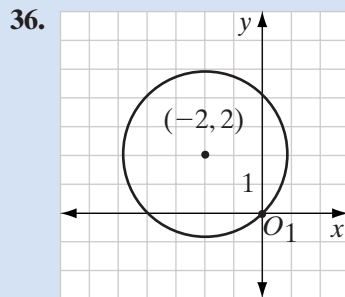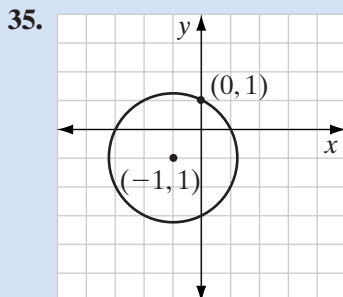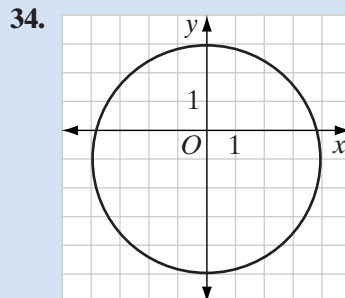**25.** $f(x) = x^2$ and $g(x) = x + 2$　　　　**26.** $f(x) = \sqrt{x}$ and $g(x) = 5x - 3$

27. $f(x) = \frac{x}{2}$ and $g(x) = 2 + 3x$　　　　**28.** $f(x) = \frac{5}{x}$ and $g(x) = 10x$

29. $f(x) = 2x + 7$ and $g(x) = \frac{x - 7}{2}$　　　　**30.** $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$

In 31–36, write the equation of each circle.

31.

32.

33.

34.

35.

36.
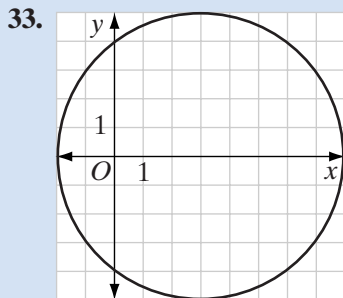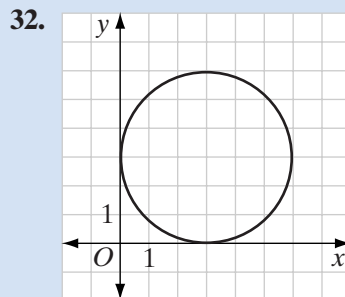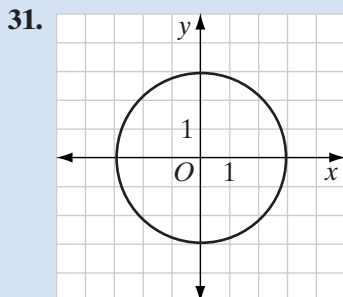
37. The graph of the circle
$(x - 2)^2 + (y + 1)^2 = 10$ and the line
$y = x - 1$ are shown in the diagram. Find
the common solutions of the equations and
check your answer.
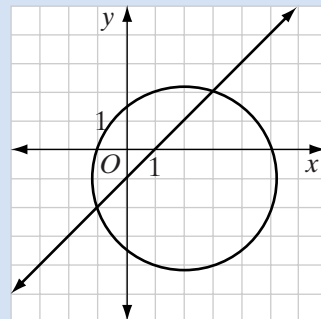
38. The graph of $y = x^2$ is shifted 4 units to the right and 2 units down. Write
the equation of the new function.

39. The graph of $y = |x|$ is stretched in the vertical direction by a factor of 3. Write an equation of the new function.

40. The graph of $y = 2x + 3$ is reflected in the x-axis. Write an equation of the new function.

41. The graph of $y = |x|$ is reflected in the x-axis and then moved 1 unit to the left and 3 units up. Write an equation of the new function.

In 42–45:

  a. Write an equation of the relation formed by interchanging the elements of the ordered pairs of each function.

  b. Determine if the relation written in **a** is an inverse function of the given function. Explain why or why not.

  c. If the given function does not have an inverse function, determine, if possible, a domain of the given function for which an inverse function exists.

42. $f(x) = 2x + 8$ **43.** $f(x) = \frac{3}{x}$

44. $f(x) = x^2$ **45.** $f(x) = \sqrt{x}$

Exploration

For this activity, use Styrofoam cones.

The shapes obtained by cutting a cone are called **conic sections**. The equation of each of these curves is a relation.

  1. What is the shape of the edge of the base of a cone? Cut the cone parallel to the base. What is the shape of the edges of the two cut surfaces?

  2. Cut the cone at an angle so that the cut does not intersect the base of the cone but is not parallel to the base of the cone. What is the shape of the edges of the cut surfaces?

  3. Cut the cone parallel to one slant edge of the cone. What is the shape of the curved portion of the edges of the cut surfaces?

  4. Cut the cone perpendicular to the base of the cone but not through a diameter of the base. Repeat this cut on a second cone and place the two cones tip to tip. What is the shape of the edges of the cut surfaces?

  5. Cut the cone perpendicular to the base of the cone through a diameter of the base. Repeat this cut on a second cone and place the two cones tip to tip. What is the shape of the edges of the cut surfaces?

Part I

Answer all questions in this part. Each correct answer will receive 2 credits. No partial credit will be allowed.

1. Which of the following is *not* a real number?

(1) -1 (3) $\sqrt{-3}$

(2) $\sqrt{3}$ (4) $0.101001000100001\ldots$

2. In simplest form, $2x(4x^2 - 5) - (7x^3 + 3x)$ is equal to

(1) $x^3 - 7x$ (3) $x^3 + 3x - 5$

(2) $x^3 - 13x$ (4) $x^3 - 3x - 10$

3. The factors of $3x^2 - 7x + 2$ are

(1) $(3x - 2)(x - 1)$ (3) $(3x + 2)(x + 1)$

(2) $(3x - 1)(x - 2)$ (4) $(3x + 1)(x + 2)$

4. The fraction $\frac{3}{\sqrt{3}}$ is equal to

(1) $\frac{\sqrt{3}}{3}$ (2) $\sqrt{3}$ (3) $3\sqrt{3}$ (4) 3

5. The solution set of $|x + 2| < 5$ is

(1) $\{x : x < 3\}$ (3) $\{x : -7 < x < 3\}$

(2) $\{x : x < -7\}$ (4) $\{x : x < -7 \text{ or } x > 3\}$

6. The solution set of $x^2 - 5x = 6$ is

(1) $\{3, -2\}$ (2) $\{-3, 2\}$ (3) $\{6, 1\}$ (4) $\{6, -1\}$

7. In simplest form, the sum $\sqrt{75} + \sqrt{27}$ is

(1) $\sqrt{102}$ (2) $34\sqrt{3}$ (3) $8\sqrt{6}$ (4) $8\sqrt{3}$

8. Which of the following is *not* a one-to-one function?

(1) $y = x + 2$ (3) $y = \frac{1}{x}$

(2) $y = |x + 2|$ (4) $y = \frac{x}{2}$

9. For which of the following are x and y inversely proportional?

(1) $y = 4x$ (2) $y = \frac{x}{4}$ (3) $y = x^2$ (4) $y = \frac{4}{x}$

10. The fraction $\frac{a - 3}{a^2 - 4}$ is undefined for

(1) $a = 0$ (3) $a = 4$

(2) $a = 3$ (4) $a = 2$ and $a = -2$

Part II

Answer all questions in this part. Each correct answer will receive 2 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

11. Factor completely: $2x^3 + 2x^2 - 2x - 2$.

12. Write $\frac{3 + \sqrt{5}}{3 - \sqrt{5}}$ as an equivalent fraction with a rational denominator.

Part III

Answer all questions in this part. Each correct answer will receive 4 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

13. Find the solution set of the inequality $x^2 - 2x - 15 < 0$.

14. Write in standard form the equation of the circle with center at $(2, -3)$ and radius 4.

Part IV

Answer all questions in this part. Each correct answer will receive 6 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

15. a. Sketch the graph of the polynomial function $y = 4x^2 - 8x - 5$.

    b. Write an equation of the axis of symmetry.

    c. What are the coordinates of the turning point?

    d. What are the zeros of the function?

16. Simplify the rational expression $\dfrac{2 + \frac{2}{a}}{1 - \frac{1}{a^2}}$ and list all values of a for which the fraction is undefined.