REAL NUMBERS AND RADICALS

Traditionally, we use units such as inches or meters to measure length, but we could technically measure with any sized unit. Two line segments are called *commeasurable* if there exists some unit with which both segments have integral measures. Therefore, the ratio of the lengths of the line segments is the ratio of integers, a rational number. Two line segments are *incommensurable* if there is no unit, no matter how small, with which both segments can have integral measures. Therefore, the ratio of the lengths is not a rational number, that is, the ratio is an irrational number.

The oldest known proof of the existence of incommensurable line segments is found in the tenth book of Euclid's *Elements* although the proof was known long before Euclid's time. Euclid established that the diagonal of a square and a side of a square are incommensurable. The ratio of their lengths is an irrational number. In arithmetic terms, what Euclid proved was that $\sqrt{2}$ was irrational. The concept of a real number, then, had its beginnings as the ratio of the lengths of line segment.

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3-1 THE REAL NUMBERS AND ABSOLUTE VALUE

In Chapter 2 we learned that a number is rational if and only if it can be written as an infinitely repeating decimal. However, we know that there are infinite decimal numbers that do not repeat. For example, the following numbers have patterns that continue infinitely but do not repeat.

$$0.20200200020000200000\ldots \qquad 0.123456789101112131415\ldots$$

These numbers are called **irrational numbers**.

DEFINITION

An **irrational number** is an infinite decimal number that does not repeat.

Other examples of irrational numbers are numbers that are the square root of an integer that is not a perfect square. For example, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, and $\sqrt{7}$ are all irrational numbers. The ratio of the circumference of a circle to its diameter, π, is also an irrational number.

DEFINITION

The union of the set of rational numbers and the set of irrational numbers is the set of **real numbers**.

Every point on the number line is associated with one and only one real number and every real number is associated with one and only one point on the number line. We say that there is a one-to-one correspondence between the real numbers and the points on the number line.

EXAMPLE I

Which of the following numbers is irrational?

$$-7, \quad \tfrac{5}{4}, \quad 2.1\overline{54}, \quad 0.12131415\ldots$$

Solution $-7 = \frac{-7}{1}$ and $\frac{5}{4}$ are each the ratio of integers and are therefore rational.

$2.1\overline{54} = 2.1545454\ldots$ is an infinite repeating decimal and is therefore rational.

$0.12131415\ldots$ is an infinite decimal that does not repeat and is therefore irrational.

**Answer** $0.12131415\ldots$

Graphing Inequalities on the Number Line

In the set of integers, the solution set of an inequality can often be listed.

For example, when the domain is the set of integers, the solution set of the inequality shown at the right is $\{-1, 0, 1, 2, 3, 4, 5\}$.

$$5 < x + 7 \qquad \leq 12$$
$$5 - 7 < x + 7 - 7 \leq 12 - 7$$
$$-2 < x \qquad\quad \leq 5$$

If the domain is the set of real numbers, the elements of the solution set cannot be listed but can be shown on the number line.
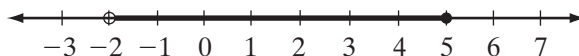

Note 1: The open circle at -2 indicates that this is a lower boundary of the set but not an element of the set. The darkened circle at 5 indicates that 5 is the upper boundary of the set and an element of the set.

Note 2: The set can be symbolized using **interval notation** as $(-2, 5]$. The curved parenthesis, (, at the left indicates that the lower boundary is *not* an element of the solution set and the bracket,], at the right indicates that the upper boundary is included in the set. The set includes all real numbers from -2 to 5, including 5 but not including -2.
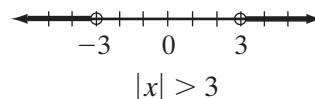
Absolute Value Inequalities

We know that if $|x| = 3$, $x = 3$, or $x = -3$. These two points on the real number line separate the line into three segments.
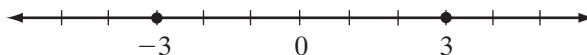
- The points between -3 and 3 are closer to 0 than are -3 and 3 and therefore have absolute values that are less than 3.

$|x| < 3$
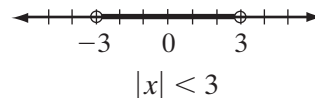
- The points to the left of -3 and to the right of 3 are farther from 0 than are -3 and 3 and therefore have absolute values that are greater than 3.

$|x| > 3$

In Chapter 1, we learned that for any *positive* value of k:

- if $|x| < k$, then $-k < x < k$.
- if $|x| > k$, then $x > k$ or $x < -k$.

    What about when k is negative? We can use the definition of absolute value to solve inequalities of the form $|x| < k$ and $|x| > k$.

Solve $|x| < k$ for negative k.	Solve $|x| > k$ for negative k.
Since $|x|$ is always non-negative and a non-negative number is always greater than a negative number, there is no value of x for which $|x| < k$ is true. The solution set is the empty set.	Since $|x|$ is always non-negative and a non-negative number is always greater than a negative number, $|x| > k$ is always true. The solution set is the set of real numbers.

SUMMARY

| Inequality	Value of k	Solution Set
$|x| < k$	$k > 0$	$\{x : -k < x < k\}$
$|x| < k$	$k < 0$	$\varnothing$
$|x| > k$	$k > 0$	$\{x : x < -k \text{ or } x > k\}$
$|x| > k$	$k < 0$	$\{$All real numbers$\}$

EXAMPLE 2

Solve for x and graph the solution set: $|3a - 1| < 5$.

Solution For a positive value of k, the solution of $|x| < k$ is $-k < x < k$.
Let $x = 3a - 1$ and $k = 5$.

> *How to Proceed*

(1) Write the solution in terms of a: $\qquad -5 < \quad 3a - 1 \quad < 5$

(2) Add 1 to each member of the inequality:

$$-5 + 1 < 3a - 1 + 1 < 5 + 1$$
$$-4 < 3a \qquad\quad < 6$$

(3) Divide each member by 3:

$$-\tfrac{4}{3} < \tfrac{3a}{3} \qquad < \tfrac{6}{3}$$
$$-\tfrac{4}{3} < a \qquad\quad < 2$$

(4) Graph the solution set:

$$-\tfrac{4}{3} \qquad 0 \qquad 2$$
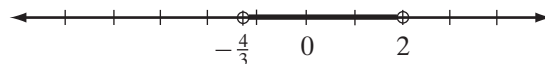
Check We can check the solution on the graphing calculator.

Graph $Y_1 = |3X - 1|$ and $Y_2 = 5$. Absolute value is entered in the calculator by using the abs (function (MATH ▶ 1). By pressing TRACE -4 ÷ 3 ENTER and TRACE 2 ENTER, we can see that the two functions intersect at points for which $x = -\tfrac{4}{3}$ and $x = 2$. From the graph, we can verify that $\left(-\tfrac{4}{3}, 2\right)$ is the solution to the inequality since the graph of Y_1 lies below the graph of Y_2 in this interval.

Y1=abs(3X-1)

X=-1.333333   Y=5

Y1=abs(3X-1)

X=2   Y=5
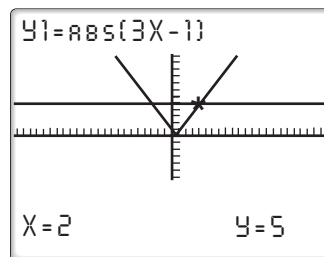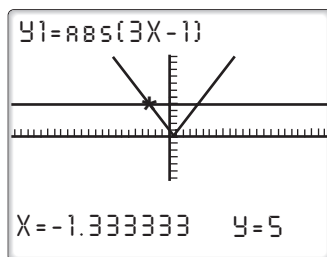
Writing About Mathematics

1. Tony said that $\dfrac{3}{1 - \frac{1}{5}}$ is irrational because it is not the ratio of integers and is therefore not a

rational number. Do you agree with Tony? Explain why or why not.

2. Maria said that since the solution of the inequality $|2x - 5| < 3$ can be found by using $-3 < 2x - 5 < 3$, then the solution of the inequality $|2x - 5| > 3$ can be found by using $-3 > 2x - 5 > 3$. Do you agree with Maria? Explain why or why not.

Developing Skills

In 3–14, determine whether each of the numbers is rational or irrational.

3. $\frac{0}{4}$

4. $2.17273747\ldots$

5. 3π

6. $\sqrt{17}$

7. $1\frac{3}{4}$

8. $\frac{\sqrt{2}}{2}$

9. $\sqrt{3} + 5$

10. $\frac{\sqrt{5}}{\sqrt{5}}$

11. $\frac{\sqrt{16}}{2}$

12. $0 + \pi$

13. $\frac{\pi}{\pi}$

14. $2\frac{1}{3} + \sqrt{3}$

In 15–26, find and graph the solution set of each inequality.

15. $|x| < 7$

16. $|a - 5| \geq 3$

17. $|2y + 5| > 9$

18. $|2 - 4b| \leq 6$

19. $|-5 - a| > 4$

20. $9 - |3x + 3| > 0$

21. $\left|5x - \frac{1}{2}\right| - \frac{3}{2} > 0$

22. $2|x + 2| \leq -3$

23. $2|x + 2| > -3$

24. $\left|\frac{5}{2}x + 2\right| \leq 0$

25. $\left|x + \frac{1}{2}\right| + 1 > \frac{1}{2}$

26. $3|2x - 2| + 2 \geq -5$

Applying Skills

27. The temperature on Mars roughly satisfies the inequality $|t - 75| \leq 145$ where t is the temperature in Fahrenheit. What is the range of temperatures on Mars?

28. Elevation of land in the United States is given by the inequality $|h - 10{,}019| \leq 10{,}301$ where h is the height in feet. What is the range of elevations in the United States?

3-2 ROOTS AND RADICALS

Square Root

We know that $4 = 2 \cdot 2$. Therefore, 4 is the square of 2, and 2 is a square root of 4. We also know that $4 = -2 \cdot (-2)$. Therefore, 4 is the square of -2, and -2 is a square root of 4. We write this as $4 = 2^2$ and $4 = (-2)^2$. Because there are two numbers whose square is 4, 4 has two square roots, written $\sqrt{4} = 2$ and $-\sqrt{4} = -2$. We can write these last two equalities as $\pm\sqrt{4} = \pm 2$.

DEFINITION
A **square root** of k is one of the two equal factors whose product is k.

Every positive real number has two square roots: a positive real number and its opposite, a negative real number. For example:

$$x^2 = 9 \qquad\qquad y^2 = 5$$
$$x = \pm\sqrt{9} = \pm 3 \qquad\qquad y = \pm\sqrt{5}$$
$$x = 3 \text{ or } x = -3 \qquad\qquad y = \sqrt{5} \text{ or } y = -\sqrt{5}$$

The **principal square root** of a positive real number is its positive square root. Thus, the principal square root of 9 is 3 and the principal square root of 5 is $\sqrt{5}$. In general, when referring to the square root of a number, we mean the principal square root.

When $\sqrt{5}$ is entered, a calculator will return the approximate decimal equivalent of $\sqrt{5}$ to as many decimal places as allowed by the calculator.

ENTER: [2nd] [√] 5 [ENTER]

DISPLAY:
```
√(5
        2.236067977
```

If we enter the decimal 2.236067977 into the calculator and square it, the calculator will return 4.999999998, an approximate value correct to nine decimal places. If we square 2.236067977 using pencil and paper we obtain a number with eighteen decimal places with 9 in the last decimal place. The exact decimal value of $\sqrt{5}$ is infinite and non-repeating.

Cube Root

DEFINITION
The **cube root** of k is one of the three equal factors whose product is k.

The cube root of k is written as $\sqrt[3]{k}$. For example, since $5 \cdot 5 \cdot 5 = 125$, 5 is one of the three equal factors whose product is 125 and $\sqrt[3]{125} = 5$. There is only one real number that is the cube root of 125. This real number is called the **principal cube root**. In a later chapter, we will enlarge the set of real numbers to form a set called the *complex numbers*. In the set of complex numbers, every real number has three cube roots, only one of which is a real number.

Note that $(-5) \cdot (-5) \cdot (-5) = -125$. Therefore, $-5 = \sqrt[3]{-125}$ and -5 is the principal cube root of -125.

The *n*th Root of a Number

DEFINITION

The ***n*th root** of k is one of the n equal factors whose product is k.

The nth root of k is written as $\sqrt[n]{k}$. In this notation, k is the **radicand**, n is the **index**, and $\sqrt[n]{k}$ is the **radical**. If k is positive, the **principal *n*th root** of k is the positive nth root.

If k is negative and n is odd, the **principal *n*th root** of k is negative. If k is negative and n is even, there is no **principal *n*th root** of k in the set of real numbers.

For example, in the expression $\sqrt[4]{16}$:

$$\text{Radical} \longrightarrow \overset{\displaystyle \ulcorner \text{Index}}{\underset{\displaystyle \llcorner \text{Radicand}}{\sqrt[4]{16}}}$$
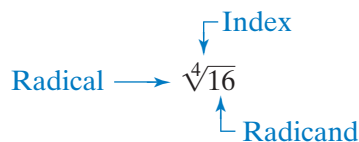
Since $2 \cdot 2 \cdot 2 \cdot 2 = 16$, $\sqrt[4]{16} = 2$ and 2 is the principal fourth root of 16. Note that $(-2) \cdot (-2) \cdot (-2) \cdot (-2) = 16$, so $-\sqrt[4]{16} = -2$. The real number 16 has two fourth roots that are real numbers: 2 (the principal fourth root) and -2.

If the real number k is not the product of n equal factors that are rational numbers, then the principal nth root of k^n, $\sqrt[n]{k}$, is irrational. For example, since 13 is a prime and has only itself and 1 as factors, the principal nth root of 13, $\sqrt[n]{13}$, is an irrational number.

Since the product of an even number of positive factors is positive and the product of an even number of negative factors is positive, $\sqrt[n]{k}$ is a real number when n is even if and only if k is positive. For example $\sqrt[4]{81} = 3$ but $\sqrt[4]{-81}$ has no real roots.

To evaluate a radical on the calculator, we need to access the MATH menu. Entry 4, $\sqrt[3]{}($, is used to evaluate cube root. Entry 5, $\sqrt[x]{}$, is used to evaluate a radical with any index. When using entry 5, the index of the root must be entered first.

<div align="center">

Evaluate $\sqrt[3]{12}$ *Evaluate* $\sqrt[5]{35}$

</div>

ENTER: **MATH** **4** 12 **ENTER**   ENTER: 5 **MATH** **5** 35 **ENTER**

DISPLAY:
```
³√(12
        2.289428485
```

DISPLAY:
```
5ˣ√35
        2.036168005
```

When a variable appears in the radicand, the radical can be simplified if the exponent of the variable is divisible by the index. For instance, since $8x^6 = (2x^2)(2x^2)(2x^2)$, $\sqrt[3]{8x^6} = 2x^2$.

EXAMPLE I

If $a^2 = 169$, find all values of a.

Solution   Since $13 \cdot 13 = 169$ and $-13 \cdot (-13) = 169$, $a = \pm\sqrt{169} = \pm13$. *Answer*

EXAMPLE 2

Evaluate each of the following in the set of real numbers:

a. $\sqrt{49}$ **b.** $-\sqrt{121b^4}$ **c.** $\sqrt[3]{-\frac{1}{8}}$ **d.** $\sqrt[4]{-81}$

Solution   **a.** $7 \cdot 7 = 49$. Therefore, 7 is the principal square root and $\sqrt{49} = 7$.

b. $(-11b^2)(-11b^2) = 121b^4$. Therefore, $11b^2$ is the principal square root of $\sqrt{121b^4}$ and $-\sqrt{121b^4} = -11b^2$.

c. $\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) = -\frac{1}{8}$. Therefore, $-\frac{1}{2}$ is the principal cube root of $-\frac{1}{8}$ and $\sqrt[3]{-\frac{1}{8}} = -\frac{1}{2}$.

d. A negative real number does not have four equal factors that are real numbers. $\sqrt[4]{-81}$ does not represent a real number.

Answers   **a.** 7   **b.** $-11b^2$ **c.** $-\frac{1}{2}$ **d.** not a real number

EXAMPLE 3

Find the length of the longer leg of a right triangle if the measure of the shorter leg is 9 centimeters and the measure of the hypotenuse is 41 centimeters.

Solution Use the Pythagorean Theorem, $a^2 + b^2 = c^2$. Let a be the length of the longer leg, b be the length of the shorter leg, and c be the length of the hypotenuse.

$$a^2 + b^2 = c^2$$
$$a^2 + 9^2 = 41^2$$
$$a^2 + 81 = 1,681$$
$$a^2 = 1,600$$
$$a = \pm 40$$

Reject the negative root.

Answer The length of the longer leg is 40 centimeters.

Exercises

Writing About Mathematics

1. a. Kevin said that if the index of a radical is even and the radicand is positive, then the radical has two real roots. Do you agree with Kevin? Explain why or why not.

   b. Kevin said that if the index of a radical is odd, then the radical has one real root and that the root is positive if the radicand is positive and negative if the radicand is negative. Do you agree with Kevin? Explain why or why not.

2. a. Sarah said that in the set of real numbers, $\sqrt{a}$ is one of the two equal factors whose product is a. Therefore, $\sqrt{a} \cdot \sqrt{a} = a$ for some values of a. Do you agree with Sarah? Explain why or why not.

   b. If you agree with Sarah, for which values of a is the statement true? Explain.

Developing Skills

In 3–10, tell whether each represents a number that is rational, irrational, or neither.

3. $\sqrt{25}$ **4.** $\sqrt{8}$ **5.** $\sqrt{-8}$ **6.** $\sqrt[3]{-8}$

7. $\sqrt{0}$ **8.** $\sqrt[4]{16}$ **9.** $\sqrt[5]{-243}$ **10.** $\sqrt{0.25}$

In 11–38, evaluate each expression in the set of real numbers.

11. $\sqrt{16}$ **12.** $\pm\sqrt{16}$ **13.** $-\sqrt{16}$ **14.** $\sqrt{625}$

15. $\sqrt{169}$ **16.** $-\sqrt{0.04}$ **17.** $\pm\sqrt{0.64}$ **18.** $\sqrt{1.44}$

19. $\sqrt[3]{27}$ **20.** $\sqrt[4]{16}$ **21.** $\sqrt[3]{-125}$ **22.** $-\sqrt[3]{125}$

23. $-\sqrt[3]{-125}$ **24.** $\sqrt[5]{-1}$ **25.** $\sqrt{\frac{4}{25}}$ **26.** $-\sqrt{\frac{49}{36}}$

27. $\sqrt[3]{\frac{8}{27}}$ **28.** $\sqrt[5]{-\frac{1}{32}}$ **29.** $\sqrt[3]{0.001}$ **30.** $\sqrt[4]{0.0256}$

31. $\sqrt{x^6}$, $x \geq 0$ **32.** $\sqrt{100c^4}$, $c \geq 0$ **33.** $\sqrt{0.25x^2}$, $x \geq 0$ **34.** $\sqrt[3]{\frac{1,000}{a^3}}$, $a \neq 0$

35. $-\sqrt[4]{\frac{b^8}{1,296}}$, $b \geq 0$ **36.** $\sqrt[5]{-0.00001y^5}$, $y \geq 0$

37. $\sqrt[k]{1}$, k an integer greater than 2   **38.** $\sqrt[k]{x^{2k}}$, k an integer greater than 2

In 39–42, find the set of real numbers for which the given radical is a real number.

39. $\sqrt{x - 2}$ **40.** $\sqrt{9 - 3x}$ **41.** $\sqrt{4x + 12}$ **42.** $\sqrt[4]{x + 5}$

In 43–46, solve each equation for the variable.

43. $x^2 = 81$ **44.** $a^2 = 196$ **45.** $b^2 = 100$ **46.** $y^2 - 169 = 0$

Applying Skills

47. The area of a square is 14 square centimeters. What is the length of a side of the square?

48. Find the length of the hypotenuse of a right triangle if the length of the longer leg is 20 feet and the length of the shorter leg is 12 feet.

49. Find the length of the shorter leg of a right triangle if the length of the longer leg is 36 inches and the length of the hypotenuse is 39 inches.

50. What is the length of a side of a square if the length of a diagonal is $\sqrt{72}$ inches?

51. The length l, width w, and height h, of a rectangular carton are $l = 12$ feet, $w = 4$ feet, and $h = 3$ feet. The length of a diagonal, d, the distance from one corner of the box to the opposite corner, is given by the formula $d^2 = l^2 + w^2 + h^2$. What is the length of the diagonal of the carton?

3-3 SIMPLIFYING RADICALS

We know that $\sqrt{4 \cdot 9} = \sqrt{36} = 6$ and that $\sqrt{4} \cdot \sqrt{9} = 2 \cdot 3 = 6$. Therefore,

$$\sqrt{4 \cdot 9} = \sqrt{4} \cdot \sqrt{9}.$$

We can state this relationship in general terms. For all non-negative real numbers a and b:

$$\boldsymbol{\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}}$$

This relationship can be used when $\sqrt{a}$ and $\sqrt{b}$ are rational numbers and when they are irrational numbers. When $\sqrt{a \cdot b}$ is an irrational number, we can often write it in simpler form. For instance:

$$\sqrt{450} \qquad\qquad \sqrt{450} \qquad\qquad \sqrt{450}$$
$$= \sqrt{9} \cdot \sqrt{50} \qquad = \sqrt{25} \cdot \sqrt{18} \qquad = \sqrt{225} \cdot \sqrt{2}$$
$$= 3\sqrt{50} \qquad\qquad = 5\sqrt{18} \qquad\qquad = 15\sqrt{2}$$

Each of these irrational numbers is equal to $\sqrt{450}$. We say that $15\sqrt{2}$ is the **simplest form** of $\sqrt{450}$ because the radicand has no factor that is a perfect square. (Alternatively, a radical is in simplest form when the radicand is a *prime number* or the product of prime numbers with each prime factor occurring once.) Note that the radicand in each of the other forms of $\sqrt{450}$ has a factor that is a perfect square and can be simplified.

$$\sqrt{450} = 3\sqrt{50} \qquad\qquad \sqrt{450} = 5\sqrt{18}$$
$$= 3\sqrt{25}\sqrt{2} \qquad\qquad\quad = 5\sqrt{9}\sqrt{2}$$
$$= 3 \cdot 5\sqrt{2} \qquad\qquad\quad = 5 \cdot 3\sqrt{2}$$
$$= 15\sqrt{2} \qquad\qquad\qquad = 15\sqrt{2}$$

We know that $x^3 \cdot x^3 = x^6$. Then for $x \geq 0$, $\sqrt{x^6} = x^3$. In general, $x^n \cdot x^n = x^{2n}$. Therefore, for $x \geq 0$:

$$\sqrt{x^{2n}} = x^n$$

In particular, when $n = 1$, we have that $\sqrt{x^2} = x$.
We can also write x^5 as $x^4 \cdot x$. Therefore,

$$\sqrt{x^5} = \sqrt{x^4 \cdot x} \qquad \sqrt{12y^7} = \sqrt{4y^6} \cdot \sqrt{3y} \qquad \sqrt{50a^3b^5} = \sqrt{25a^2b^4} \cdot \sqrt{2ab}$$
$$= \sqrt{x^4} \cdot \sqrt{x} \qquad\quad = 2y^3\sqrt{3y} \qquad\qquad\quad = 5ab^2\sqrt{2ab}$$
$$= x^2\sqrt{x} \qquad\qquad\quad (y \geq 0) \qquad\qquad\qquad (a \geq 0 \text{ and } b \geq 0)$$
$$(x \geq 0)$$

Fractional Radicands

When the radicand of an irrational number is a fraction, the radical is in simplest form when it is written with an integer under the radical sign. For example, $\sqrt{\frac{2}{3}}$ does not have a perfect square factor in either the numerator or the denominator of the radicand. The radical is not in simplest form. To simplify this radical, write $\frac{2}{3}$ as an equivalent fraction with a denominator that is a perfect square.

$$\sqrt{\tfrac{2}{3}} = \sqrt{\tfrac{2}{3} \times \tfrac{3}{3}} = \sqrt{\tfrac{6}{9}} = \sqrt{\tfrac{1}{9} \times \tfrac{6}{1}} = \sqrt{\tfrac{1}{9}} \times \sqrt{6} = \tfrac{1}{3}\sqrt{6} \ \text{ or } \ \tfrac{\sqrt{6}}{3}$$

Note that since $\frac{a}{c} = \left(\frac{1}{c}\right)\left(\frac{a}{1}\right) = \left(\frac{1}{c}\right)(a)$ and the square root of a product is equal to the product of the square roots of the factors:

$$\sqrt{\tfrac{a}{c}} = \sqrt{\tfrac{1}{c} \cdot \tfrac{a}{1}} = \sqrt{\tfrac{1}{c}} \cdot \sqrt{\tfrac{a}{1}} = \tfrac{1}{\sqrt{c}}\sqrt{a} = \tfrac{\sqrt{a}}{\sqrt{c}}$$

Therefore, for any non-negative a and positive c:

$$\sqrt{\tfrac{a}{c}} = \tfrac{\sqrt{a}}{\sqrt{c}}$$

This leads to an alternative method of simplifying a fractional radicand:

$$\sqrt{\tfrac{5}{7}} = \tfrac{\sqrt{5}}{\sqrt{7}} \times \tfrac{\sqrt{7}}{\sqrt{7}} = \tfrac{\sqrt{35}}{\sqrt{49}} = \tfrac{\sqrt{35}}{7}$$

or

$$\sqrt{\tfrac{a}{c}} = \tfrac{\sqrt{a}}{\sqrt{c}} \times \tfrac{\sqrt{c}}{\sqrt{c}} = \tfrac{\sqrt{ac}}{\sqrt{c^2}} = \tfrac{\sqrt{ac}}{c}$$

EXAMPLE 1

Write each expression in simplest form.

a. $\sqrt{\tfrac{8}{9}}$ b. $\sqrt{\tfrac{4}{5}}$ c. $\sqrt{\tfrac{3}{8}}$

Solution   a. Here the denominator is a perfect square and 8 has a perfect square factor, 4:

$$\sqrt{\tfrac{8}{9}} = \sqrt{\tfrac{4}{9} \times 2} = \sqrt{\tfrac{4}{9}} \times \sqrt{2} = \tfrac{2}{3}\sqrt{2} \; \textit{Answer}$$

b. Here the numerator is a perfect square but the denominator is not. We must multiply by $\tfrac{5}{5}$ to write the radicand as a fraction with a denominator that is a perfect square:

$$\sqrt{\tfrac{4}{5}} = \sqrt{\tfrac{4}{5} \times \tfrac{5}{5}} = \sqrt{\tfrac{20}{25}} = \sqrt{\tfrac{4}{25} \times 5} = \sqrt{\tfrac{4}{25}} \times \sqrt{5} = \tfrac{2}{5}\sqrt{5} \; \textit{Answer}$$

c. Here neither the numerator nor the denominator is a perfect square. We must first write the radicand as a fraction whose denominator is a perfect square. Since 16 is the smallest perfect square that is a multiple of 8, we can multiply by $\tfrac{2}{2}$ to obtain a radicand with a denominator that is a perfect square, 16:

$$\sqrt{\tfrac{3}{8}} = \sqrt{\tfrac{3}{8} \times \tfrac{2}{2}} = \sqrt{\tfrac{6}{16}} = \tfrac{\sqrt{6}}{\sqrt{16}} = \tfrac{\sqrt{6}}{4} \; \textit{Answer}$$

Note: Part **b** could also be solved by an alternative method. Rewrite the radical as the quotient of two radicals. Multiply by $\tfrac{\sqrt{5}}{\sqrt{5}}$. Then simplify:

$$\sqrt{\tfrac{4}{5}} = \tfrac{\sqrt{4}}{\sqrt{5}} \times \tfrac{\sqrt{5}}{\sqrt{5}}$$
$$= \tfrac{\sqrt{4}\sqrt{5}}{\sqrt{25}}$$
$$= \tfrac{2\sqrt{5}}{5} \; \textit{Answer}$$

To write square root radicals with a variable in the denominator, we want the variable in the denominator to be a perfect square, that is, to have an exponent that is even. For example, simplify $\sqrt{\frac{9a}{8b^3}}$ if $a > 0$ and $b > 0$. To do this we must write the radicand of the denominator as a perfect square. The smallest perfect square that is a multiple of 8 is 8×2 or 16 and the smallest perfect square that is a multiple of b^3 is $b^3 \cdot b$ or b^4. In order to have a denominator of $\sqrt{16b^4}$ we must multiply the given radical by $\sqrt{\frac{2b}{2b}}$.

$$\sqrt{\frac{9a}{8b^3}} = \sqrt{\frac{9a}{8b^3} \cdot \frac{2b}{2b}} = \sqrt{\frac{18ab}{16b^4}} = \sqrt{\frac{9}{16b^4}} \cdot \sqrt{2ab} = \frac{3}{4b^2}\sqrt{2ab}$$

Alternatively, we could rewrite the radical as the quotient of radicals, multiply by $\frac{\sqrt{8b^3}}{\sqrt{8b^3}}$, and then simplify:

$$\sqrt{\frac{9a}{8b^3}} = \frac{\sqrt{9a}}{\sqrt{8b^3}} \cdot \frac{\sqrt{8b^3}}{\sqrt{8b^3}} = \frac{\sqrt{72ab^3}}{\sqrt{(8b^3)^2}} = \frac{6b\sqrt{2ab}}{8b^3} = \frac{3\sqrt{2ab}}{4b^2}$$

Roots with Index *n*

The rules that we have derived for square roots are true for roots with any index.

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b} \quad \textbf{and} \quad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

In order to write $\sqrt[n]{k}$ in simplest form, we must find the largest factor of k of the form a^n. For instance:

$$\sqrt[3]{40} = \sqrt[3]{8 \times 5} = \sqrt[3]{8} \times \sqrt[3]{5} = 2\sqrt[3]{5}$$

Recall that if the index of a root is n, then if a is a rational number, $\sqrt[n]{a^k}$ is a rational number when k is divisible by n. For example:

$$\sqrt[3]{a^6} = a^2 \qquad \sqrt[4]{b^{12}} = b^3 \qquad \sqrt[5]{q^5} = q^1 = q \qquad \sqrt[n]{x^{an}} = x^a$$

In order to write $\sqrt[3]{x^8}$ in simplest form, find the largest multiple of 3 that is less than 8. That multiple is 6. Write x^8 as $x^6 \cdot x^2$.

$$\sqrt[3]{x^8} = \sqrt[3]{x^6} \cdot \sqrt[3]{x^2} = x^2 \cdot \sqrt[3]{x^2}$$

In order to simplify $\sqrt[n]{\frac{p}{q}}$, we must write $\frac{p}{q}$ as an equivalent fraction whose denominator is of the form r^n. For instance:

$$\sqrt[4]{\frac{1}{2}} = \sqrt[4]{\frac{1}{2} \times \frac{2^3}{2^3}} = \sqrt[4]{\frac{2^3}{2^4}} = \frac{\sqrt[4]{2^3}}{\sqrt[4]{2^4}} = \frac{\sqrt[4]{8}}{2} \text{ or } \frac{1}{2}\sqrt[4]{8}$$

$$\sqrt[4]{\frac{3a}{2b^3c^2}} = \sqrt[4]{\frac{3a}{2b^3c^2} \cdot \frac{8bc^2}{8bc^2}} = \sqrt[4]{\frac{24abc^2}{16b^4c^4}}$$

$$= \sqrt[4]{\frac{1}{16b^4c^4}}\sqrt[4]{24abc^2}$$

$$= \frac{1}{2bc}\sqrt[4]{24abc^2}$$

EXAMPLE 1

Write each of the following in simplest radical form:

a. $\sqrt{48}$ **b.** $\sqrt{\frac{9}{20b}}$ **c.** $\sqrt[3]{54y^8}$

Solution **a.** The factors of 48 are 2×24, 3×16, 4×12, and 6×8. The pair of factors with the largest perfect square is 3×16:

$$\sqrt{48} = \sqrt{16 \times 3} = \sqrt{16} \times \sqrt{3} = 4\sqrt{3} \ \text{Answer}$$

b. The smallest multiple of $20b$ that is a perfect square is $100b^2$. Multiply the radicand by $\frac{5b}{5b}$:

$$\sqrt{\frac{9}{20b}} = \sqrt{\frac{9}{20b} \cdot \frac{5b}{5b}} = \sqrt{\frac{45b}{100b^2}}$$
$$= \sqrt{\frac{9}{100b^2} \cdot 5b}$$
$$= \sqrt{\frac{9}{100b^2}} \cdot \sqrt{5b}$$
$$= \frac{3}{10b}\sqrt{5b} \ \text{Answer}$$

c. The factors of 54 are 2×27, 3×18, and 6×9. The factor 27 is the cube of 3. The largest multiple of the index that is less than the exponent of y is 6:

$$\sqrt[3]{54y^8} = \sqrt[3]{27y^6 \cdot 2y^2} = \sqrt[3]{27y^6} \cdot \sqrt[3]{2y^2} = 3y^2 \cdot \sqrt[3]{2y^2} \ \text{Answer}$$

EXAMPLE 2

a. Write $2\sqrt{50b^3}$ in simplest radical form.

b. For what values b does $2\sqrt{50b^3}$ represent a real number?

Solution **a.** $2\sqrt{50b^3} = 2\sqrt{25b^2} \cdot \sqrt{2b}$ **b.** $2\sqrt{50b^3}$ is a real number when

$\qquad\qquad = 2(5b)\sqrt{2b}$ $50b^3$ is positive or 0, that is, for

$\qquad\qquad = 10b\sqrt{2b} \ \text{Answer}$ $b \geq 0$. *Answer*

EXAMPLE 3

Write $\sqrt{0.2}$ in simplest radical form.

Solution **METHOD 1**
Write 0.2 as the product of a perfect square and a prime:

$$\sqrt{0.2} = \sqrt{0.20}$$
$$= \sqrt{0.04 \times 5}$$
$$= \sqrt{0.04} \times \sqrt{5}$$
$$= 0.2\sqrt{5}$$

METHOD 2

Write 0.2 as a common fraction, $\frac{2}{10}$. Write $\frac{2}{10}$ as an equivalent fraction with a denominator that is a perfect square, 100:

$$\sqrt{0.2} = \sqrt{\tfrac{2}{10}}$$
$$= \sqrt{\tfrac{2}{10} \times \tfrac{10}{10}}$$
$$= \sqrt{\tfrac{20}{100}}$$
$$= \sqrt{\tfrac{4}{100}} \times \sqrt{5}$$
$$= \tfrac{2}{10}\sqrt{5}$$

Answer $0.2\sqrt{5}$ or $\tfrac{2}{10}\sqrt{5}$

Note: For a decimal fraction, the denominator will be a perfect square when there are two decimal places (hundredths), four decimal places (ten-thousandths) or any even number of decimal places (a denominator of 10^{2n} for n a counting number). For example, $\sqrt{0.01} = 0.1$, $\sqrt{0.0001} = 0.01$, $\sqrt{0.000001} = 0.001$.

Exercises

Writing About Mathematics

1. Explain the difference between $-\sqrt{36}$ and $\sqrt{-36}$.

2. If a is a negative number, is $-\sqrt[3]{-8a^3}$ a positive number, a negative number, or not a real number? Explain your answer.

Developing Skills

In 3–38, write each radical in simplest radical form. Variables in the radicand of an even index are non-negative. Variables occurring in the denominator of a fraction are non-zero.

3. $\sqrt{12}$

4. $\sqrt{50}$

5. $\sqrt{32}$

6. $\sqrt{8b^3}$

7. $\sqrt{98c^4}$

8. $3\sqrt{20y^5}$

9. $5\sqrt{200xy^2}$

10. $4\sqrt{363x^5y^7}$

11. $\frac{1}{2}\sqrt{72ab^5}$

12. $\sqrt[3]{16}$

13. $\sqrt[3]{24}$

14. $\sqrt[3]{40a^4}$

15. $\sqrt[3]{375x^5y^6}$

16. $\sqrt[4]{48a^9b^3}$

17. $\sqrt{\frac{4a^4}{25}}$

18. $\sqrt{\frac{3b^3}{49}}$

19. $\sqrt{\frac{6}{81y^6}}$

20. $\sqrt{\frac{a^5}{2}}$

21. $\sqrt{\frac{a^3}{5b}}$

22. $\sqrt{\frac{1}{6xy}}$

23. $\sqrt{\frac{3x}{4y}}$

24. $\sqrt{\frac{3}{5b^5}}$

25. $\sqrt{\frac{5a}{18}}$

26. $\sqrt{\frac{15}{8b^3}}$

27. $\sqrt{\dfrac{4a^3}{27b^3}}$ **28.** $\sqrt{\dfrac{9}{50xy^7}}$ **29.** $\sqrt{\dfrac{8x}{5y^3}}$ **30.** $\sqrt{0.08}$

31. $\sqrt{0.5}$ **32.** $\sqrt{2.42ab^5}$ **33.** $\sqrt{0.001x^3}$ **34.** $\sqrt{128x^2}$

35. $\sqrt{300c}$ **36.** $\sqrt[3]{\dfrac{a^3}{3}}$ **37.** $\sqrt[4]{\dfrac{2a^8}{b^2c^3}}$ **38.** $\sqrt[4]{32x^5y^4}$

Applying Skills

39. The lengths of the legs of a right triangle are 8 centimeters and 12 centimeters. Express the length of the hypotenuse in simplest radical form.

40. The length of one leg of an isosceles right triangle is 6 inches. Express the length of the hypotenuse in simplest radical form.

41. The length of the hypotenuse of a right triangle is 24 meters and the length of one leg is 12 meters. Express the length of the other leg in simplest radical form.

42. The dimensions of a rectangle are 15 feet by 10 feet. Express the length of the diagonal in simplest radical form.

43. The area of a square is 150 square feet. Express the length of a side of the square in simplest radical form.

44. The area of a circular pool is $5x^2y^4\pi$ square meters. Express the radius of the pool in simplest radical form.

45. The area of a triangle is $\sqrt[5]{243x^5y^{10}}$ square units. If the length of a side of the triangle is $\sqrt[5]{x^5}$, express the length of the altitude to that side in simplest radical form.

46. Tom has a trunk that is 18 inches wide, 32 inches long, and 16 inches high. Which of the following objects could Tom store in the trunk? (There may be more than one answer.)

(1) A walking stick 42 inches long

(2) A fishing rod 37 inches long

(3) A yardstick

(4) A baseball bat 34 inches long

3-4 ADDING AND SUBTRACTING RADICALS

Recall that the sum or difference of fractions or of similar algebraic terms is found by using the distributive property.

$$\frac{2}{5} + \frac{4}{5} = \frac{1}{5} \times 2 + \frac{1}{5} \times 4 \qquad\qquad 2x + 4x = x(2 + 4)$$
$$= \frac{1}{5}(2 + 4) \qquad\qquad\qquad\qquad = x(6)$$
$$= \frac{1}{5}(6) \qquad\qquad\qquad\qquad\quad = 6x$$
$$= \frac{6}{5}$$

We do not write all of the steps shown above when adding fractions or algebraic terms. The principles that justify each of these steps assure us that the results are correct.

We can apply the same principles to the addition or subtraction of radical expressions. To express the sum or difference of two radicals as a single radical, the radicals must have the same index and the same radicand, that is, they must be **like radicals**. We can use the same procedure that we use to add or subtract like terms.

$$2\sqrt{5} + 4\sqrt{5} \qquad\qquad 4\sqrt{3b} - 3\sqrt{3b} \qquad\qquad \sqrt[3]{2} + 7\sqrt[3]{2}$$
$$= \sqrt{5}(2 + 4) \qquad\qquad = \sqrt{3b}(4 - 3) \qquad\qquad = \sqrt[3]{2}(1 + 7)$$
$$= \sqrt{5}(6) \qquad\qquad = \sqrt{3b}(1) \qquad\qquad = \sqrt[3]{2}(8)$$
$$= 6\sqrt{5} \qquad\qquad = \sqrt{3b} \qquad\qquad = 8\sqrt[3]{2}$$

Two radicals that do not have the same radicand or do not have the same index are **unlike radicals**. The sum or difference of two unlike radicals cannot be expressed as a single radical if they cannot be written as equivalent like radicals. Each of the following, $\sqrt{2} + \sqrt{3}$, $\sqrt[3]{2} + \sqrt{2}$, and $\sqrt[4]{2} + \sqrt{3}$, are the sums of unlike radicals and are in simplest radical form.

Simplifying Unlike Radicals

The fractions $\frac{1}{3}$ and $\frac{1}{4}$ do not have common denominators but we can add these two fractions by writing them as equivalent fractions with a common denominator. The radicals $\sqrt{8}$ and $\sqrt{50}$ have the same index but do not have the same radicand. We can add these radicals if they can be written in simplest form with a common radicand.

$$\sqrt{8} = \sqrt{4}\sqrt{2} = 2\sqrt{2} \quad\text{and}\quad \sqrt{50} = \sqrt{25}\sqrt{2} = 5\sqrt{2}$$
$$\sqrt{8} + \sqrt{50} = 2\sqrt{2} + 5\sqrt{2} = 7\sqrt{2}$$
$$\sqrt{12b^3} = \sqrt{4b^2}\sqrt{3b} = 2b\sqrt{3b} \quad\text{and}\quad \sqrt{27b^3} = \sqrt{9b^2}\sqrt{3b} = 3b\sqrt{3b}$$
$$\sqrt{12b^3} + \sqrt{27b^3} = 2b\sqrt{3b} + 3b\sqrt{3b} = 5b\sqrt{3b}$$

Note that it is not always possible to express two unlike radicals as like radicals. For example, the sum $\sqrt{2} + \sqrt{3}$ cannot be written with the same radicand and therefore the sum cannot be written as one radical.

EXAMPLE I

Write $\sqrt{12} + \sqrt{20} - \sqrt{45} + \sqrt{\frac{1}{3}}$ in simplest form.

Solution Simplify each radical and combine like radicals.

$$\sqrt{12} + \sqrt{20} - \sqrt{45} + \sqrt{\frac{1}{3}} = \sqrt{4}\sqrt{3} + \sqrt{4}\sqrt{5} - \sqrt{9}\sqrt{5} + \sqrt{\frac{3}{9}}$$
$$= 2\sqrt{3} + 2\sqrt{5} - 3\sqrt{5} + \tfrac{1}{3}\sqrt{3}$$
$$= 2\sqrt{3} + \tfrac{1}{3}\sqrt{3} + 2\sqrt{5} - 3\sqrt{5}$$
$$= \tfrac{7}{3}\sqrt{3} - \sqrt{5} \ \textit{Answer}$$

EXAMPLE 2

Simplify: $3x\sqrt{\frac{1}{3x}} + \sqrt{300x}$

Solution Simplify each radical.

$$3x\sqrt{\frac{1}{3x}} = 3x\sqrt{\frac{1}{3x} \cdot \frac{3x}{3x}} = 3x\sqrt{\frac{3x}{9x^2}} = 3x\frac{\sqrt{3x}}{3x} = 3\overset{1}{x}\frac{\sqrt{3x}}{\underset{1}{3x}} = \sqrt{3x}$$
$$\sqrt{300x} = \sqrt{100}\sqrt{3x} = 10\sqrt{3x}$$

Add the simplified radicals.

$$3x\sqrt{\frac{1}{3x}} + \sqrt{300x} = \sqrt{3x} + 10\sqrt{3x} = 11\sqrt{3x} \ (x > 0) \ \textit{Answer}$$

EXAMPLE 3

Solve and check: $4x - \sqrt{8} = \sqrt{72}$

Solution

$$4x - \sqrt{8} = \sqrt{72}$$
$$4x - \sqrt{8} + \sqrt{8} = \sqrt{72} + \sqrt{8}$$
$$4x = \sqrt{36}(\sqrt{2}) + \sqrt{4}(\sqrt{2})$$
$$4x = 6\sqrt{2} + 2\sqrt{2}$$
$$4x = 8\sqrt{2}$$
$$\frac{4x}{4} = \frac{8\sqrt{2}}{4}$$
$$x = 2\sqrt{2}$$

Check
$$4x - \sqrt{8} = \sqrt{72}$$
$$4(\mathbf{2\sqrt{2}}) - \sqrt{4}(\sqrt{2}) \overset{?}{=} \sqrt{36}(\sqrt{2})$$
$$8\sqrt{2} - 2\sqrt{2} \overset{?}{=} 6\sqrt{2}$$
$$6\sqrt{2} = 6\sqrt{2} \ \checkmark$$

Answer $x = 2\sqrt{2}$

Exercises

Writing About Mathematics

1. Danielle said that $3x\sqrt{\frac{1}{3x}}$ could be simplified by writing $3x\sqrt{\frac{1}{3x}}$ as $\sqrt{\frac{9x^2}{3x}} = \sqrt{3x}$. Do you agree with Danielle? Justify your answer.

2. Does $\sqrt{16} + \sqrt{48} = \sqrt{64}$? Justify your answer.

Developing Skills

In 3–38 write each expression in simplest form. Variables in the radicand with an even index are non-negative. Variables occurring in the denominator of a fraction are non-zero.

3. $\sqrt{2} + 5\sqrt{2}$

4. $6\sqrt{5} - 4\sqrt{5}$

5. $8\sqrt{3} + \sqrt{3}$

6. $5\sqrt{7} - \sqrt{7}$

7. $\sqrt{50} + \sqrt{2}$

8. $3\sqrt{5y} - \sqrt{20y}$

9. $\sqrt{250a^2} + \sqrt{10a^2}$

10. $8\sqrt{11b^4} - \sqrt{99b^4}$

11. $\sqrt{24xy^2} + \sqrt{54xy^2}$

12. $\sqrt{200a^7} - \sqrt{50a^7}$

13. $\sqrt{98c^5} - \sqrt{18c^5}$

14. $x\sqrt{32x} + \sqrt{128x^3}$

15. $4b\sqrt{24b^3} + \sqrt{54b^5}$

16. $3x^3\sqrt{80} + 2\sqrt{125x^6}$

17. $\sqrt{5} + \sqrt{\frac{1}{5}}$

18. $\sqrt{24} + 2\sqrt{\frac{3}{2}}$

19. $14\sqrt{\frac{1}{7}} + \sqrt{28}$

20. $\sqrt{\frac{1}{2x}} + \sqrt{\frac{1}{2x}}$

21. $a\sqrt{45} + \sqrt{20a^2} - 5\sqrt{2a}$

22. $x\sqrt{600} - 2\sqrt{24x^2} + 4x\sqrt{96}$

23. $2\sqrt{3y} - 5y^2 + 4\sqrt{3y} + \sqrt{36y^4}$

24. $\sqrt{162a^4b^3} + 3 - ab\sqrt{18a^2b} - 1$

25. $\sqrt{12} - \sqrt{24} + \sqrt{48} + \sqrt{27}$

26. $5\sqrt{\frac{1}{5}} - \sqrt{\frac{1}{10}} + \sqrt{20}$

27. $\sqrt{\frac{1}{6}} + \sqrt{\frac{8}{3}} - \sqrt{\frac{2}{3}}$

28. $\sqrt[3]{2} + \sqrt[3]{16}$

29. $\sqrt[3]{54} + \sqrt[3]{128}$

30. $\sqrt[4]{48} - \sqrt[4]{3}$

31. $\sqrt{9x} + \sqrt{25x}$

32. $\sqrt{100y} - \sqrt{25y}$

33. $\sqrt{8a} - \sqrt{2a}$

34. $\sqrt{18b^2} + \sqrt{800b^2}$

35. $\sqrt{63a^2} - \sqrt{45a^2}$

36. $\sqrt{4ab^2} - \sqrt{ab^2}$

37. $\sqrt{50x^3} + \sqrt{200x^3}$

38. $\sqrt{49x^3} - 2x\sqrt{4x}$

In 39–42, solve and check each equation.

39. $5x - \sqrt{3} = \sqrt{48}$

40. $12y + \sqrt{32} = \sqrt{200}$

41. $4a + \sqrt{6} = a + \sqrt{96}$

42. $y + \sqrt{20} = \sqrt{45} - 2y$

Applying Skills

In 43–47, express each answer in simplest radical form.

43. The lengths of the sides of a triangle are $\sqrt{75}$ inches, $\sqrt{27}$ inches, and $\sqrt{108}$ inches. What is the perimeter of the triangle?

44. The length of each of the two congruent sides of an isosceles triangle is $\sqrt{500}$ feet and the length of the third side is $\sqrt{45}$ feet. What is the perimeter of the triangle?

45. The lengths of the legs of a right triangle are $\sqrt{18}$ centimeters and $\sqrt{32}$ centimeters.

a. Find the length of the hypotenuse.

b. Find the perimeter of the triangle.

46. The length of each leg of an isosceles right triangle is $\sqrt{98}$ inches.

a. Find the length of the hypotenuse.

b. What is the perimeter of the triangle?

47. The dimensions of a rectangle are $\sqrt{250}$ meters and $\sqrt{1,440}$ meters.

a. Express the perimeter of the rectangle in simplest radical form.

b. Express the length of the diagonal of the rectangle in simplest radical form.

3-5 MULTIPLYING RADICALS

Recall that if a and b are non-negative numbers, $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$. Therefore, by the symmetric property of equality, we can say that $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$. Recall also that for any positive number a, $\sqrt{a} \cdot \sqrt{a} = \sqrt{a^2} = a$. We can use these rules to multiply radicals.

For example:

$$\sqrt{4} \cdot \sqrt{25} = \sqrt{100} = 10$$

$$\sqrt{2} \cdot \sqrt{2} = \sqrt{4} = 2$$

$$\sqrt{8} \cdot \sqrt{2} = (\sqrt{4} \cdot \sqrt{2}) \cdot \sqrt{2} = 2(\sqrt{2} \cdot \sqrt{2}) = 2(2) = 4$$

$$\sqrt{6a^3} \cdot \sqrt{18a} = \sqrt{108a^4} = \sqrt{36a^4} \cdot \sqrt{3} = 6a^2\sqrt{3} \quad (a \geq 0)$$

Note: $\sqrt{-2} \times \sqrt{-8} \neq \sqrt{16}$ because $\sqrt{-2}$ and $\sqrt{-8}$ are not real numbers.

The rule for the multiplication of radicals is true for any two radicals with the same index if the radicals are real numbers. That is, since $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$ is true for non-negative real numbers, the following is also true by the symmetric property of equality:

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab} \ (a, b \geq 0)$$

Note: This rule does not apply when the index n is even and a or b are negative. For instance, $\sqrt[4]{-2} \times \sqrt[4]{-8} \neq \sqrt[4]{16}$ because $\sqrt[4]{-2}$ and $\sqrt[4]{-8}$ are not real numbers.

EXAMPLE I

Simplify: *Answers*

a. $\sqrt[3]{8} \cdot \sqrt[3]{27}$ $= \sqrt[3]{216} = 6$

or

$= 2 \cdot 3 = 6$

b. $\sqrt[4]{48x^2} \cdot \sqrt[4]{\frac{x^2}{3}}$ $= \sqrt[4]{48x^2 \cdot \frac{x^2}{3}} = \sqrt[4]{16x^4} = 2x \ (x \geq 0)$

Multiplying Sums That Contain Radicals

The distributive property for multiplication over addition or subtraction is true for all real numbers. Therefore, we can apply it to irrational numbers that contain radicals.

For example:

$$\sqrt{3}(2 + \sqrt{3}) = \sqrt{3}(2) + \sqrt{3}(\sqrt{3}) = 2\sqrt{3} + 3$$

$$(2 + \sqrt{5})(1 + \sqrt{5}) = 2(1 + \sqrt{5}) + \sqrt{5}(1 + \sqrt{5})$$
$$= 2 + 2\sqrt{5} + \sqrt{5} + 5 = 7 + 3\sqrt{5}$$

$$(3 + \sqrt{2})(3 - \sqrt{2}) = 3(3 - \sqrt{2}) + \sqrt{2}(3 - \sqrt{2})$$
$$= 9 - 3\sqrt{2} + 3\sqrt{2} - 2 = 7$$

In each of these examples, we are multiplying irrational numbers. We know that $\sqrt{3}$, $\sqrt{5}$ and $\sqrt{2}$ are all irrational numbers. The sum of a rational number and an irrational number is always an irrational number. Therefore, $2 + \sqrt{3}$, $2 + \sqrt{5}$, $1 + \sqrt{5}$, $3 + \sqrt{2}$, and $3 - \sqrt{2}$ are all irrational numbers. The product of irrational numbers may be a rational or an irrational number.

EXAMPLE 2

Express each of the following products in simplest form:

a. $\sqrt{5}(\sqrt{10})$ **b.** $(3 + \sqrt{6a})(1 + \sqrt{2a})$ **c.** $(\sqrt{5} + \sqrt{7})(\sqrt{5} - \sqrt{7})$

Solution **a.** $\sqrt{5}(\sqrt{10}) = \sqrt{50} = \sqrt{25} \cdot \sqrt{2} = 5\sqrt{2}$ *Answer*

b. $(3 + \sqrt{6a})(1 + \sqrt{2a}) = 3(1 + \sqrt{2a}) + \sqrt{6a}(1 + \sqrt{2a})$

$$= 3 + 3\sqrt{2a} + \sqrt{6a} + \sqrt{12a^2}$$

$$= 3 + 3\sqrt{2a} + \sqrt{6a} + \sqrt{4a^2}(\sqrt{3})$$

$$= 3 + 3\sqrt{2a} + \sqrt{6a} + 2a\sqrt{3} \ \text{\textit{Answer}}$$

c. $(\sqrt{5} + \sqrt{7})(\sqrt{5} - \sqrt{7}) = \sqrt{5}(\sqrt{5} - \sqrt{7}) + \sqrt{7}(\sqrt{5} - \sqrt{7})$

$$= 5 - \sqrt{35} + \sqrt{35} - 7$$

$$= -2 \ \text{\textit{Answer}}$$

EXAMPLE 3

The length of a side, s, of an equilateral triangle is $\sqrt{5}$ inches and the length of the altitude, h, is $\sqrt{\frac{15}{4}}$ inches. Find the area of the triangle.

Solution   Area of a triangle $= \frac{1}{2}bh$

$$= \frac{1}{2}(\sqrt{5})\left(\sqrt{\frac{15}{4}}\right)$$

$$= \frac{1}{2}\left(\sqrt{\frac{75}{4}}\right)$$

$$= \frac{1}{2}\left(\sqrt{\frac{25}{4}} \cdot \sqrt{3}\right)$$

$$= \frac{1}{2}\left(\frac{5}{2}\sqrt{3}\right)$$

$$= \frac{5}{4}\sqrt{3} \ \text{sq inches} \ \text{\textit{Answer}}$$

Exercises

Writing About Mathematics

1. Brandon said that if a is a positive real number, then $3a$, $4a$, and $5a$ are the lengths of the sides of a right triangle. Therefore, $3\sqrt{2}$, $4\sqrt{2}$, and $5\sqrt{2}$ are the lengths of the sides of a right triangle. Do you agree with Brandon? Justify your answer.

2. Jennifer said that if a is a positive real number, then $\sqrt[4]{a^2} = \sqrt{a}$. Do you agree with Jennifer? Justify your answer.

Developing Skills

In 3–41, express each product in simplest form. Variables in the radicand with an even index are non-negative.

3. $\sqrt{2} \cdot \sqrt{8}$ **4.** $\sqrt{5} \cdot \sqrt{45}$ **5.** $\sqrt{3} \cdot \sqrt{27}$

6. $\sqrt{8} \cdot \sqrt{12}$ **7.** $-\sqrt{10} \cdot \sqrt{18}$ **8.** $3\sqrt{2} \cdot \sqrt{10}$

9. $\sqrt{\frac{1}{3}} \cdot \sqrt{24}$ **10.** $\sqrt{21} \cdot \sqrt{\frac{4}{3}}$ **11.** $8\sqrt{6} \cdot \sqrt{\frac{5}{12}}$

12. $\left(\sqrt{12}\right)^2$ **13.** $\left(3\sqrt{3}\right)^2$ **14.** $\left(-2\sqrt{5}\right)^2$

15. $\sqrt{x^3} \cdot \sqrt{4x}$ **16.** $2\sqrt{ab} \cdot 2\sqrt{ab^2}$ **17.** $\sqrt{5y} \cdot \sqrt{4y^3}$

18. $\sqrt{x^5 y^3} \cdot \sqrt{3xy}$ **19.** $7\sqrt{a} \cdot 5\sqrt{\frac{a}{9}}$ **20.** $\sqrt{\frac{x}{2}} \cdot \sqrt{\frac{x^2}{2}}$

21. $\sqrt{\frac{a}{3}} \cdot \sqrt{\frac{a^2}{4}}$ **22.** $\sqrt[3]{2} \cdot \sqrt[3]{4}$ **23.** $\sqrt[3]{15a^2} \cdot \sqrt[3]{9a^4}$

24. $\sqrt[4]{27} \cdot \sqrt[4]{3}$ **25.** $\sqrt{2}\left(2 + \sqrt{2}\right)$ **26.** $\sqrt{5}\left(1 - \sqrt{10}\right)$

27. $\sqrt{8}\left(6 + \sqrt{2}\right)$ **28.** $\sqrt{5a}\left(\sqrt{5a} - 3\right)$ **29.** $\sqrt{12xy^3}\left(\sqrt{3xy} + 3\right)$

30. $\left(1 + \sqrt{5}\right)\left(3 - \sqrt{5}\right)$ **31.** $\left(9 + \sqrt{2b}\right)\left(1 + \sqrt{2b}\right)$ **32.** $\left(7 + \sqrt{5y}\right)\left(3 - \sqrt{5y}\right)$

33. $\left(7 + \sqrt{5b}\right)\left(7 - \sqrt{5b}\right)$ **34.** $\left(x - \sqrt[4]{3y}\right)\left(2x - \sqrt[4]{3y}\right)$ **35.** $\left(\sqrt{6} + 6\right)\left(\sqrt{6} - 7\right)$

36. $\left(\sqrt{6} + 6c\right)\left(\sqrt{6} - 6c\right)$ **37.** $\left(a + \sqrt{b}\right)\left(a - \sqrt{b}\right)$ **38.** $\left(1 - \sqrt{3}\right)^2$

39. $\left(3 + \sqrt{5ab^3}\right)^2$ **40.** $\left(1 - \sqrt{7}\right)\left(1 + \sqrt{7}\right)\left(1 + \sqrt{7}\right)$ **41.** $\left(2 - \sqrt{5}\right)\left(2 + \sqrt{5}\right)^2$

Applying Skills

42. The length of a side of a square is $48\sqrt{2}$ meters. Express the area of the square in simplest form.

43. The dimensions of a rectangle are $12\sqrt{2}$ feet by $\sqrt{50}$ feet. Express the area of the rectangle in simplest form.

44. The dimensions of a rectangular solid are $\sqrt{5}$ inches by $\left(2 + \sqrt{3}\right)$ inches by $\left(2 - \sqrt{3}\right)$ inches. Express the volume of the solid in simplest form.

45. The lengths of the legs of a right triangle in feet are $\left(3 + \sqrt{3}\right)$ and $\left(3 - \sqrt{3}\right)$.

    a. Find the length of the hypotenuse of the triangle.

    b. Express, in simplest form, the perimeter of the triangle.

    c. What is the area of the triangle?

46. The radius of the surface of a circular pool is $\left(2 + \sqrt{xy^5}\right)$ meters. Express the area of the pool in simplest form.

3-6 DIVIDING RADICALS

We can use what we know about the relationship between division and multiplication and the rule for the multiplication of radicals to write a rule for the division of radicals. We know that for any non-negative a and positive b, $\sqrt{a} \div \sqrt{b} = \frac{\sqrt{a}}{\sqrt{b}}$. Therefore, if a and b are positive real numbers, then:

$$\sqrt{a} \div \sqrt{b} = \sqrt{a} \cdot \sqrt{\tfrac{1}{b}} = \sqrt{\tfrac{a}{b}} \quad \text{or} \quad \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\tfrac{a}{b}}$$

Recall that a similar rule is true for roots with any index, and so we can write:

$$\sqrt[n]{a} \div \sqrt[n]{b} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\tfrac{a}{b}}$$

Note: $\sqrt[3]{\tfrac{-8}{27}} = \frac{\sqrt[3]{-8}}{\sqrt[3]{27}} = \frac{-2}{3} = -\tfrac{2}{3}$ is a true statement because $\sqrt[3]{\tfrac{-8}{27}}$, $\sqrt[3]{-8}$, and $\sqrt[3]{27}$ are all real numbers. However, we cannot write $\sqrt{\tfrac{-4}{9}} = \frac{\sqrt{-4}}{\sqrt{9}}$ because $\sqrt{\tfrac{-4}{9}}$ and $\sqrt{-4}$ are not real numbers.

EXAMPLE I

Express $\sqrt{27a} \div \sqrt{3a^3}$ in simplest radical form when a is a positive real number.

Solution

$$\sqrt{27a} \div \sqrt{3a^3} = \frac{\sqrt{27a}}{\sqrt{3a^3}} = \sqrt{\frac{27a}{3a^3}} = \sqrt{\frac{9}{a^2} \cdot \frac{\overset{1}{\cancel{3a}}}{\cancel{3a}}} = \sqrt{\frac{9}{a^2}} = \frac{3}{a} \quad \text{Answer}$$

Alternative Solution

$$\sqrt{27a} \div \sqrt{3a^3} = \frac{\sqrt{27a}}{\sqrt{3a^3}} = \frac{\sqrt{9}(\sqrt{3a})}{\sqrt{a^2}(\sqrt{3a})} = \frac{3\overset{1}{\cancel{\sqrt{3a}}}}{a\cancel{\sqrt{3a}}} = \frac{3}{a} \quad \text{Answer}$$

EXAMPLE 2

Write $\frac{\sqrt{15}}{\sqrt{45}}$ in simplest radical form.

Solution

$$\frac{\sqrt{15}}{\sqrt{45}} = \sqrt{\frac{15}{45}} = \sqrt{\frac{1}{3}} = \sqrt{\frac{1}{3} \times \frac{3}{3}} = \sqrt{\frac{3}{9}} = \frac{\sqrt{3}}{3} \quad \text{Answer}$$

Alternative Solution

$$\frac{\sqrt{15}}{\sqrt{45}} = \frac{\sqrt{3} \times \sqrt{5}}{\sqrt{9} \times \sqrt{5}} = \frac{\sqrt{3}}{3} \times 1 = \frac{\sqrt{3}}{3} \quad \text{Answer}$$

EXAMPLE 3

Simplify: $\dfrac{12\sqrt{10a} + 8\sqrt{15b}}{4\sqrt{5ab}}$

Solution Use the distributive property of division over addition.

$$\dfrac{12\sqrt{10a} + 8\sqrt{15b}}{4\sqrt{5ab}} = \dfrac{12\sqrt{10a}}{4\sqrt{5ab}} + \dfrac{8\sqrt{15b}}{4\sqrt{5ab}}$$

$$= 3\sqrt{\tfrac{2}{b}} + 2\sqrt{\tfrac{3}{a}}$$

$$= 3\sqrt{\tfrac{2}{b} \cdot \tfrac{b}{b}} + 2\sqrt{\tfrac{3}{4} \cdot \tfrac{a}{a}}$$

$$= 3\sqrt{\tfrac{2b}{b^2}} + 2\sqrt{\tfrac{3a}{a^2}}$$

$$= \tfrac{3}{b}\sqrt{2b} + \tfrac{2}{a}\sqrt{3a} \quad (a > 0, b > 0) \quad \textit{Answer}$$

Exercises

Writing About Mathematics

1. Jonathan said that $\dfrac{\sqrt{10}}{2} = \sqrt{5}$. Do you agree with Jonathan? Justify your answer.

2. Show that the quotient of two irrational numbers can be either rational or irrational.

Developing Skills

In 3–29 write each quotient in simplest form. Variables in the radicand with an even index are non-negative. Variables occurring in the denominator of a fraction are non-zero.

3. $\sqrt{24} \div \sqrt{6}$

4. $\sqrt{75} \div \sqrt{3}$

5. $\sqrt{72} \div \sqrt{8}$

6. $\sqrt{50a^3} \div \sqrt{5a}$

7. $\sqrt{24x^2} \div \sqrt{3x^3}$

8. $\dfrac{\sqrt{150}}{\sqrt{3}}$

9. $\dfrac{\sqrt{54}}{\sqrt{2}}$

10. $\dfrac{\sqrt{300}}{\sqrt{25}}$

11. $\dfrac{\sqrt{35a^3}}{\sqrt{10a}}$

12. $\dfrac{\sqrt{80x^2y}}{\sqrt{30xy^2}}$

13. $\dfrac{\sqrt{27b}}{\sqrt{6b^2}}$

14. $\dfrac{3}{\sqrt{3x}}$

15. $\dfrac{7}{\sqrt{7y}}$

16. $\dfrac{\sqrt{12a^2}}{\sqrt{4a}}$

17. $\dfrac{\sqrt{18c^3}}{\sqrt{9c}}$

18. $\dfrac{4\sqrt{2} + 8\sqrt{12}}{2\sqrt{2}}$

19. $\dfrac{3\sqrt{10} - 9\sqrt{50}}{3\sqrt{5}}$

20. $\dfrac{\sqrt{72} + \sqrt{54}}{\sqrt{18}}$

21. $\dfrac{\sqrt{20} - \sqrt{5}}{\sqrt{5}}$

22. $\dfrac{\sqrt{48} + \sqrt{3}}{\sqrt{3}}$

23. $\dfrac{\sqrt{10} + \sqrt{15}}{\sqrt{10}}$

24. $\dfrac{5 + 6\sqrt{5}}{\sqrt{5}}$

25. $\dfrac{\sqrt[3]{27x^3} + \sqrt[3]{36x^5}}{\sqrt[3]{3x^3}}$

26. $\dfrac{\sqrt[4]{ab^4}}{\sqrt[4]{a^2b^4}}$

27. $\dfrac{\sqrt[4]{c^6}}{\sqrt[5]{c^5}}$

28. $\dfrac{\sqrt[3]{24w^2}}{\sqrt[3]{3w^4}}$

29. $\dfrac{\sqrt{64x^4} + \sqrt[4]{40x^6}}{\sqrt[4]{x^6}}$

Applying Skills

30. The area of a rectangle is $25\sqrt{35}$ square feet and the width is $10\sqrt{5}$ feet. Find the length of the rectangle in simplest radical form.

31. The area of a right triangle is $6\sqrt{2}$ square centimeters and the length of one leg is $\sqrt{12}$ centimeters.

   a. What is the length of the other leg?

   b. What is the length of the hypotenuse?

3-7 RATIONALIZING A DENOMINATOR

To **rationalize the denominator** of a fraction means to write the fraction as an equivalent fraction with a denominator that is a rational number.

A fraction is in simplest form when the denominator is a positive integer, that is, a rational number. For example, we found the simplest form of a fraction such as $\frac{\sqrt{5}}{\sqrt{2}}$ by multiplying the numerator and denominator by $\sqrt{2}$ so that the denominator would be an integer.

$$\sqrt{\tfrac{5}{2}} = \sqrt{\tfrac{5}{2} \times \tfrac{2}{2}} = \sqrt{\tfrac{10}{4}} = \frac{\sqrt{10}}{\sqrt{4}} = \frac{\sqrt{10}}{2} = \tfrac{1}{2}\sqrt{10}$$

How do we write the fraction $\frac{3}{2 + \sqrt{6}}$ in simplest form, that is, with a rational denominator? To do so, we must multiply numerator and denominator of the fraction by some numerical expression that will make the denominator a rational number.

Recall that $(a + b)(a - b) = a^2 - b^2$. For example:

$$(1 + \sqrt{2})(1 - \sqrt{2}) \qquad (5 - \sqrt{3})(5 + \sqrt{3}) \qquad (2 + \sqrt{6})(2 - \sqrt{6})$$
$$= 1^2 - (\sqrt{2})^2 \qquad\quad = 5^2 - (\sqrt{3})^2 \qquad\quad = 2^2 - (\sqrt{6})^2$$
$$= 1 - 2 \qquad\qquad\quad = 25 - 3 \qquad\qquad = 4 - 6$$
$$= -1 \qquad\qquad\qquad = 22 \qquad\qquad\quad = -2$$

The binomials $(a + b)$ and $(a - b)$ are called **conjugates**. We can use this last example to simplify $\frac{3}{2 + \sqrt{6}}$. If we multiply the fraction by 1 in the form $\frac{2 - \sqrt{6}}{2 - \sqrt{6}}$, we will have a fraction whose denominator is an integer.

$$\frac{3}{2 + \sqrt{6}} = \frac{3}{2 + \sqrt{6}} \times \frac{2 - \sqrt{6}}{2 - \sqrt{6}} = \frac{6 - 3\sqrt{6}}{2^2 - (\sqrt{6})^2} = \frac{6 - 3\sqrt{6}}{4 - 6} = \frac{6 - 3\sqrt{6}}{-2} = \frac{-6 + 3\sqrt{6}}{2}$$

If we use a calculator to find a rational approximation for the given fraction and for the simplest form of the given fraction, these approximations will be equal.

ENTER: 3 [÷] [(] 2 + [2nd] [√]
6 [)] [)] [ENTER]

DISPLAY:
```
3/(2+√(6))
         .6742346142
```

ENTER: [(] −6 [+] 3 [2nd] [√]
6 [)] [)] [÷] 2 [ENTER]

DISPLAY:
```
(-6+3√(6))/2
         .6742346142
```

Recall that when the given numerical expression is written as a fraction, the line of the fraction is a grouping symbol. When entering these fractions into the calculator, the numerator or denominator that is written as a sum or a difference must be enclosed by parentheses.

To write the fraction $\dfrac{4}{2a + \sqrt{8b}}$ with a rational denominator, multiply the fraction by $\dfrac{2a - \sqrt{8b}}{2a - \sqrt{8b}}$ and simplify the result.

$$\frac{4}{2a + \sqrt{8b}} \cdot \frac{2a - \sqrt{8b}}{2a - \sqrt{8b}} = \frac{4(2a - \sqrt{8b})}{4a^2 - 8b}$$
$$= \frac{4(2a - 2\sqrt{2b})}{4(a^2 - 2b)}$$
$$= \frac{8(a - \sqrt{2b})}{4(a^2 - 2b)}$$
$$= \frac{2(a - \sqrt{2b})}{a^2 - 2b}$$

The fraction $\dfrac{2(a - \sqrt{2b})}{a^2 - 2b}$ has a rational denominator if a and b are rational numbers.

EXAMPLE I

Write $\dfrac{7}{2\sqrt{7}}$ as an equivalent fraction with a rational denominator.

Solution When $\sqrt{7}$ is multiplied by itself, the product is 7, a rational number. Multiply the fraction by 1 in the form $\dfrac{\sqrt{7}}{\sqrt{7}}$.

$$\frac{7}{2\sqrt{7}} = \frac{7}{2\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}} = \frac{7\sqrt{7}}{2(7)} = \frac{\overset{1}{7}\sqrt{7}}{2(7)} = \frac{\sqrt{7}}{2} \; \textit{Answer}$$

EXAMPLE 2

Rationalize the denominator of the fraction $\frac{3 + \sqrt{2}}{3 - \sqrt{2}}$.

Solution   The conjugate of $3 - \sqrt{2}$ is $3 + \sqrt{2}$. Multiply the fraction by 1 in the form $\frac{3 + \sqrt{2}}{3 + \sqrt{2}}$.

$$\frac{3 + \sqrt{2}}{3 - \sqrt{2}} = \frac{3 + \sqrt{2}}{3 - \sqrt{2}} \times \frac{3 + \sqrt{2}}{3 + \sqrt{2}} = \frac{9 + 3\sqrt{2} + 3\sqrt{2} + 2}{9 - 2} = \frac{11 + 6\sqrt{2}}{7} \quad Answer$$

Note: The numerator is $(3 + \sqrt{2})^2$ or $3^2 + 2(3\sqrt{2}) + (\sqrt{2})^2$.

EXAMPLE 3

Find the sum: $\frac{\sqrt{2}}{3 - \sqrt{6}} + \frac{6}{\sqrt{8}}$

Solution   Rationalize the denominator of each fraction.

$$\frac{\sqrt{2}}{3 - \sqrt{6}} = \frac{\sqrt{2}}{3 - \sqrt{6}} \times \frac{3 + \sqrt{6}}{3 + \sqrt{6}} = \frac{3\sqrt{2} + \sqrt{12}}{3^2 - (\sqrt{6})^2} = \frac{3\sqrt{2} + \sqrt{4}(\sqrt{3})}{9 - 6} = \frac{3\sqrt{2} + 2\sqrt{3}}{3}$$

$$\frac{6}{\sqrt{8}} = \frac{6}{\sqrt{8}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{6\sqrt{2}}{\sqrt{16}} = \frac{6\sqrt{2}}{4} = \frac{3\sqrt{2}}{2}$$

Add the fractions. In order to add fractions, we need a common denominator. The common denominator of $\frac{3\sqrt{2} + 2\sqrt{3}}{3}$ and $\frac{3\sqrt{2}}{2}$ is 6.

$$\begin{aligned} \frac{\sqrt{2}}{3 - \sqrt{6}} + \frac{6}{\sqrt{8}} &= \frac{3\sqrt{2} + 2\sqrt{3}}{3} \times \frac{2}{2} + \frac{3\sqrt{2}}{2} \times \frac{3}{3} \\ &= \frac{6\sqrt{2} + 4\sqrt{3}}{6} + \frac{9\sqrt{2}}{6} \\ &= \frac{6\sqrt{2} + 4\sqrt{3} + 9\sqrt{2}}{6} \\ &= \frac{15\sqrt{2} + 4\sqrt{3}}{6} \quad Answer \end{aligned}$$

EXAMPLE 4

Solve for x and check the solution: $x\sqrt{2} = 3 - x$

Solution   *How to Proceed*

(1) Write an equivalent equation with only terms in x on the left side of the equation and only constant terms on the right side. Add x to each side of the equation:

$$x\sqrt{2} = 3 - x$$
$$x\sqrt{2} + x = 3 - x + x$$
$$x\sqrt{2} + x = 3$$

(2) Factor the left side of the equation:

$$x(\sqrt{2} + 1) = 3$$

(3) Divide both sides of the equation by the coefficient of x:

$$\frac{x(\sqrt{2} + 1)}{\sqrt{2} + 1} = \frac{3}{\sqrt{2} + 1}$$

(4) Rationalize the denominator:

$$x = \frac{3}{\sqrt{2} + 1} \times \frac{\sqrt{2} - 1}{\sqrt{2} - 1}$$
$$x = \frac{3(\sqrt{2} - 1)}{2 - 1}$$
$$x = \frac{3(\sqrt{2} - 1)}{1}$$
$$x = 3\sqrt{2} - 3$$

(5) *Check:*

$$x\sqrt{2} = 3 - x$$
$$(3\sqrt{2} - 3)(\sqrt{2}) \overset{?}{=} 3 - (3\sqrt{2} - 3)$$
$$3(2) - 3\sqrt{2} \overset{?}{=} 3 - 3\sqrt{2} + 3$$
$$6 - 3\sqrt{2} = 6 - 3\sqrt{2} ✔$$

Exercises

Writing About Mathematics

1. Justin simplified $\frac{7}{2\sqrt{7}}$ by first writing 7 as $\sqrt{49}$ and then dividing numerator and denominator by $\sqrt{7}$.

   a. Show that Justin's solution is correct.

   b. Can $\frac{7}{2\sqrt{5}}$ be simplified by using the same procedure? Explain why or why not.

2. To rationalize the denominator of $\frac{4}{2 + \sqrt{8}}$, Brittany multiplied by $\frac{2 - \sqrt{8}}{2 - \sqrt{8}}$ and Justin multiplied by $\frac{1 - \sqrt{2}}{1 - \sqrt{2}}$. Explain why both are correct.

Developing Skills

In 3–38, rationalize the denominator and write each fraction in simplest form. All variables represent positive numbers.

3. $\frac{1}{\sqrt{3}}$ **4.** $\frac{5}{\sqrt{10}}$ **5.** $\frac{4}{\sqrt{2}}$ **6.** $\frac{4}{2\sqrt{3}}$

7. $\frac{15}{5\sqrt{3}}$ **8.** $\frac{4}{8\sqrt{6}}$ **9.** $\frac{12}{\sqrt{27}}$ **10.** $\frac{6}{\sqrt{12}}$

11. $\dfrac{8}{\sqrt{24}}$ **12.** $\dfrac{2\sqrt{2}}{4\sqrt{3}}$ **13.** $\dfrac{5\sqrt{5}}{15\sqrt{2}}$ **14.** $\dfrac{\sqrt{24}}{2\sqrt{6}}$

15. $\dfrac{1}{3 + \sqrt{5}}$ **16.** $\dfrac{1}{5 - \sqrt{2}}$ **17.** $\dfrac{1}{1 + \sqrt{3}}$ **18.** $\dfrac{4}{3 - \sqrt{3}}$

19. $\dfrac{3}{1 + \sqrt{5}}$ **20.** $\dfrac{4}{4 + \sqrt{7}}$ **21.** $\dfrac{3}{\sqrt{5} - 2}$ **22.** $\dfrac{9}{\sqrt{7} + 2}$

23. $\dfrac{\sqrt{2}}{2 - \sqrt{2}}$ **24.** $\dfrac{6}{3 + \sqrt{3}}$ **25.** $\dfrac{\sqrt{5x}}{\sqrt{5x} - 2}$ **26.** $\dfrac{\sqrt{20y}}{y\sqrt{5} + 1}$

27. $\dfrac{2 + \sqrt{2}}{3 - \sqrt{2}}$ **28.** $\dfrac{6 - \sqrt{7}}{4 - \sqrt{7}}$ **29.** $\dfrac{\sqrt{10} - 1}{\sqrt{10} + 1}$ **30.** $\dfrac{7 + \sqrt{5}}{7 - \sqrt{5}}$

31. $\dfrac{a + 2}{b - \sqrt{2}}$ **32.** $\dfrac{2\sqrt{x} - 5\sqrt{y}}{\sqrt{x} + \sqrt{y}}$ **33.** $\dfrac{4}{\sqrt{z} + 8}$ **34.** $\dfrac{\sqrt{a}}{\sqrt{a} - 2}$

35. $\dfrac{2}{\sqrt{x}} + \dfrac{2}{\sqrt{y}}$ **36.** $\dfrac{1}{\sqrt{x} + 6} - \dfrac{2}{\sqrt{6}}$ **37.** $\dfrac{2\sqrt{a}}{\sqrt{b}} + \dfrac{2\sqrt{b}}{\sqrt{a}}$ **38.** $\dfrac{3}{x + \sqrt{2}} + \dfrac{5}{\sqrt{x}}$

In 39–42: **a.** Write each fraction in simplest radical form. **b.** Use a calculator to find a rational approximation for the given fraction. **c.** Use a calculator to find a rational approximation for the fraction in simplest form.

39. $\dfrac{4}{\sqrt{6}}$ **40.** $\dfrac{2 + \sqrt{3}}{\sqrt{3}}$

41. $\dfrac{4}{\sqrt{3} - 1}$ **42.** $\dfrac{3 + \sqrt{7}}{3 - \sqrt{7}}$

In 43–46, solve and check each equation.

43. $2a + \sqrt{50} = \sqrt{98}$ **44.** $5x - \sqrt{12} = \sqrt{108} - 3x$

45. $y\sqrt{3} + 1 = 3 - y$ **46.** $7 - b\sqrt{8} = b\sqrt{5} + 4$

Applying Skills

47. The area of a rectangle is 24 square inches. The length of the rectangle is $\sqrt{5} + 1$ inches. Express the width of the rectangle in simplest form.

48. The perimeter of an isosceles triangle is $\sqrt{50}$ feet. The lengths of the sides are in the ratio $3 : 3 : 4$. Find the length of each side of the triangle.

3-8 SOLVING RADICAL EQUATIONS

An equation that contains at least one radical term with a variable in the radicand is called a **radical equation**. For example, $\sqrt{2x - 3} = 5$ is a radical equation. Since the radical is a square root, we can solve this equation by squaring both sides of the equation.

Solution:	*Check:*
$\sqrt{2x-3} = 5$	$\sqrt{2x-3} = 5$
$(\sqrt{2x-3})^2 = 5^2$	$\sqrt{2(14)-3} \overset{?}{=} 5$
$2x - 3 = 25$	$\sqrt{28-3} \overset{?}{=} 5$
$2x = 28$	$\sqrt{25} \overset{?}{=} 5$
$x = 14$	$5 = 5 ✔$

Note that squaring both sides of a radical equation does not always result in an equivalent equation. For example, if the given equation had been $\sqrt{2x-3} = -5$, the equation obtained by squaring both sides would have been $2x - 3 = 25$, the same as the equation obtained by squaring both sides of $\sqrt{2x-3} = 5$. The solution of $2x - 3 = 25$ is not a root of $\sqrt{2x-3} = -5$. The radical $\sqrt{2x-3}$ represents the principal root of $2x - 3$, which is a positive number. Therefore, the equation $\sqrt{2x-3} = -5$ has no real root. There is no real number such that $\sqrt{2x-3} = -5$.

Often an equation must be rewritten as an equivalent equation with the radical alone on one side of the equation before squaring both sides of the equation. For example, to solve the equation $x - \sqrt{5-x} = 3$, we must first add $-x$ to both sides of the equation to isolate the radical.

	Check: $x = 4$	Check: $x = 1$
$x - \sqrt{5-x} = 3$	$x - \sqrt{5-x} = 3$	$x - \sqrt{5-x} = 3$
$-\sqrt{5-x} = 3 - x$	$4 - \sqrt{5-4} \overset{?}{=} 3$	$1 - \sqrt{5-1} \overset{?}{=} 3$
$(-\sqrt{5-x})^2 = (3-x)^2$	$4 - \sqrt{1} \overset{?}{=} 3$	$1 - \sqrt{4} \overset{?}{=} 3$
$5 - x = 9 - 6x + x^2$	$4 - 1 \overset{?}{=} 3$	$1 - 2 \overset{?}{=} 3$
$0 = 4 - 5x + x^2$	$3 = 3 ✔$	$-1 \neq 3 ✗$
$0 = (4-x)(1-x)$		
$0 = 4 - x \mid 0 = 1 - x$		
$x = 4 \qquad \mid x = 1$		

The root of this equation is $x = 4$.

Note that after squaring both sides of the equation, we have a quadratic equation that has two real roots. One of these is the root of the given equation, $x - \sqrt{5-x} = 3$ which is equivalent to $-\sqrt{5-x} = 3 - x$. The other is the root of the equation $-\sqrt{5-x} = x - 3$. When we square both sides of either of these equations, we obtain the same equation, $5 - x = 9 - 6x + x^2$.

EXAMPLE 1

Solve and check: $4 + \sqrt{1 - 3x} = 12$

Solution

How to Proceed

(1) Isolate the radical by adding -4 to both sides of the equation:

(2) Square both sides of the equation:

$$4 + \sqrt{1 - 3x} = 12$$
$$\sqrt{1 - 3x} = 8$$
$$(\sqrt{1 - 3x})^2 = 8^2$$
$$1 - 3x = 64$$

(3) Add -1 to both sides of the equation:

(4) Divide both sides of the equation by -3:

$$-3x = 63$$
$$x = -21$$

(5) *Check:*

$$4 + \sqrt{1 - 3x} = 12$$
$$4 + \sqrt{1 - 3(-21)} \overset{?}{=} 12$$
$$4 + \sqrt{1 + 63} \overset{?}{=} 12$$
$$4 + \sqrt{64} \overset{?}{=} 12$$
$$4 + 8 \overset{?}{=} 12$$
$$12 = 12 ✔$$

Answer $x = -21$

EXAMPLE 2

Find the value of a such that $\sqrt[3]{4 - 2a} = -2$.

Solution

How to Proceed

(1) The radical is already isolated. Since the radical is a cube root, cube both sides of the equation:

$$\sqrt[3]{4 - 2a} = -2$$
$$(\sqrt[3]{4 - 2a})^3 = (-2)^3$$
$$4 - 2a = -8$$

(2) Solve the resulting equation for a:

$$-2a = -12$$
$$a = 6$$

(3) *Check:*

$$\sqrt[3]{4 - 2a} = -2$$
$$\sqrt[3]{4 - 2(6)} \overset{?}{=} -2$$
$$\sqrt[3]{4 - 12} \overset{?}{=} -2$$
$$\sqrt[3]{-8} \overset{?}{=} -2$$
$$-2 = -2 ✔$$

Answer $a = 6$

EXAMPLE 3

What is the solution set of $x = 1 + \sqrt{15 - 7x}$?

Solution

How to Proceed

(1) Write the equation with the radical alone on one side of the equation by adding -1 to both sides of the equation:

$$x = 1 + \sqrt{15 - 7x}$$
$$x - 1 = \sqrt{15 - 7x}$$

(2) Square both sides of the equation:

$$(x - 1)^2 = (\sqrt{15 - 7x})^2$$
$$x^2 - 2x + 1 = 15 - 7x$$

(3) Write the quadratic equation in standard form:

$$x^2 + 5x - 14 = 0$$

(4) Factor the polynomial.

$$(x + 7)(x - 2) = 0$$

(5) Set each factor equal to 0 and solve for x:

$$\begin{array}{c|c} x + 7 = 0 & x - 2 = 0 \\ x = -7 & x = 2 \end{array}$$

(6) *Check:*

Let $x = -7$:	Let $x = 2$:
$x = 1 + \sqrt{15 - 7x}$	$x = 1 + \sqrt{15 - 7x}$
$-7 \overset{?}{=} 1 + \sqrt{15 - 7(-7)}$	$2 \overset{?}{=} 1 + \sqrt{15 - 7(2)}$
$-7 \overset{?}{=} 1 + \sqrt{15 + 49}$	$2 \overset{?}{=} 1 + \sqrt{15 - 14}$
$-7 \overset{?}{=} 1 + \sqrt{64}$	$2 \overset{?}{=} 1 + \sqrt{1}$
$-7 \overset{?}{=} 1 + 8$	$2 \overset{?}{=} 1 + 1$
$-7 \neq 9$ ✗	$2 = 2$ ✔

-7 is not a solution of $x = 1 + \sqrt{15 - 7x}$ and 2 is a solution of $x = 1 + \sqrt{15 - 7x}$.

Answer The solution set of $x = 1 + \sqrt{15 - 7x}$ is {2}.

EXAMPLE 4

Solve the equation $\sqrt{x + 5} = 1 + \sqrt{x}$.

How to Proceed

(1) Square both sides of the equation. The right side will be a trinomial that has a radical in one of its terms:

$$(\sqrt{x + 5})^2 = (1 + \sqrt{x})^2$$
$$x + 5 = 1 + 2\sqrt{x} + x$$

(2) Simplify the equation obtained in step 1 and isolate the radical:

$$4 = 2\sqrt{x}$$

(3) Square both sides of the equation obtained in step 2:

$$4^2 = (2\sqrt{x})^2$$
$$16 = 4x$$

(4) Solve for the variable:

$$x = 4$$

(5) *Check:*

$$\sqrt{x + 5} = 1 + \sqrt{x}$$
$$\sqrt{4 + 5} \stackrel{?}{=} 1 + \sqrt{4}$$
$$3 = 3 ✔$$

Answer $x = 4$

Exercises

Writing About Mathematics

1. Explain why $\sqrt{x + 3} < 0$ has no solution in the set of real numbers while $\sqrt{x + 3} \geq 0$ is true for all real numbers greater than or equal to -3.

2. In Example 3, are $x - 1 = \sqrt{15 - 7x}$ and $(x - 1)^2 = (\sqrt{15 - 7x})^2$ equivalent equations? Explain why or why not.

Developing Skills

In 3–38, solve each equation for the variable, check, and write the solution set.

3. $\sqrt{a} = 5$

4. $\sqrt{x} = 7$

5. $4\sqrt{y} = 12$

6. $\sqrt{4y} = 12$

7. $2\sqrt{b} = 8$

8. $\sqrt{2b} = 8$

9. $1 + \sqrt{x} = 3$

10. $\sqrt{1 + x} = 3$

11. $5 + \sqrt{a} = 7$

12. $\sqrt{5 + a} = 7$

13. $3 - \sqrt{y} = 1$

14. $\sqrt{3 - y} = 1$

15. $3 - \sqrt{2x + 5} = 0$

16. $8 + \sqrt{2x - 1} = 15$

17. $\sqrt{5x + 2} = \sqrt{9x - 14}$

18. $\sqrt{20 - 2x} = \sqrt{5x - 8}$

19. $x = \sqrt{4x + 5}$

20. $y = \sqrt{y + 2}$

21. $x + 3 = \sqrt{1 - 3x}$ **22.** $a - 2 = \sqrt{2a - 1}$ **23.** $b - 3 = \sqrt{3b - 11}$

24. $3 - b = \sqrt{3b - 11}$ **25.** $x = 1 + \sqrt{x + 11}$ **26.** $x + \sqrt{x + 1} = 5$

27. $x + 4\sqrt{x} = 5$ **28.** $3 + \sqrt{4x - 3} = 2x$ **29.** $2 + \sqrt{3x - 2} = 3x$

30. $\sqrt[3]{x} = 2$ **31.** $5 + \sqrt[3]{a + 2} = 3$ **32.** $2 + \sqrt[3]{3b - 2} = 6$

33. $1 - \sqrt[3]{x + 7} = 4$ **34.** $-10 + \sqrt[4]{n - 2} = -8$ **35.** $\sqrt{x + 5} = 1 + \sqrt{x}$

36. $1 + \sqrt{2x} = \sqrt{3x + 1}$ **37.** $\sqrt{x + 4} - \sqrt{x - 1} = 1$ **38.** $\sqrt{5x} - \sqrt{x + 4} = 3$

Applying Skills

39. The lengths of the legs of an isosceles right triangle are x and $\sqrt{6x + 16}$. What are the lengths of the legs of the triangle?

40. The width of a rectangle is x and the length is $\sqrt{x - 1}$. If the width is twice the length, what are the dimensions of the rectangle?

41. In $\triangle ABC$, $AB = \sqrt{7x + 5}$, $BC = \sqrt{5x + 15}$, and $AC = \sqrt{2x}$.

a. If $AB = BC$, find the length of each side of the triangle.

b. Express the perimeter of the triangle in simplest radical form.

CHAPTER SUMMARY

An **irrational number** is an infinite decimal number that does not repeat. The union of the set of rational numbers and the set of irrational numbers is the set of **real numbers**.

If the domain is the set of real numbers, the elements of the solution set of an inequality usually cannot be listed but can be shown on the number line.

If $a \geq 0$ and $|x| = a$, then $x = a$ or $x = -a$. These two points on the real number line separate the line into three segments. The segment between a and $-a$ is the graph of the solution set of $|x| < a$ and the other two segments are the graph of the solution set of $|x| > a$.

A **square root** of a positive number is one of the two equal factors whose product is that number. The **principal square root** of a positive real number is its positive square root. The **cube root** of k is one of the three equal factors whose product is k. The **nth root** of k is one of the n equal factors whose product is k. The nth root of k is written as $\sqrt[n]{k}$ where k is the **radicand**, n is the **index**, and $\sqrt[n]{k}$ is the **radical**.

An irrational number of the form $\sqrt[n]{a}$ is in simplest form when a is an integer that has no integral factor of the form b^n.

To express the sum or difference of two radicals as a single radical, the radicals must have the same index and the same radicand, that is, they must be **like radicals**. The distributive property is used to add like radicals.

For any non-negative real numbers a and b:

$$\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$$

$$\sqrt{a} \div \sqrt{b} = \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}} \quad (b \neq 0)$$

$$\sqrt{a} \cdot \sqrt{a} = \sqrt{a^2} = a$$

For any non-negative real number a and natural number n:

$$\sqrt{a^{2n}} = a^n$$

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, then:

$$\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$$

$$\sqrt[n]{a} \div \sqrt[n]{b} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}} \quad (b \neq 0)$$

To **rationalize the denominator** of a fraction means to write the fraction as an equivalent fraction with a denominator that is a rational number.

The binomials $a + b$ and $a - b$ are conjugate binomials. When the denominator of a fraction is of the form $a \pm \sqrt{b}$ or $\sqrt{a} \pm \sqrt{b}$, we can rationalize the fraction by multiplying by a fraction whose numerator and denominator are the conjugates of the given denominator.

An equation that contains at least one radical term with a variable in the radicand is called a **radical equation**. To solve an equation that contains a square root radical, isolate the radical and square both sides of the equation. The derived equation may or may not be equivalent to the given equation and a check is necessary.

VOCABULARY

3-1 Irrational numbers • Real numbers • Interval notation

3-2 Square root • Principal square root • Cube root • nth root • Radicand • Index • Radical • Principal nth root

3-3 Simplest form

3-4 Like radicals • Unlike radicals

3-7 Rationalize the denominator • Conjugates

3-8 Radical equation

REVIEW EXERCISES

In 1–8, find and graph the solution set of each inequality.

1. $|1 - 4x| < 2$ **2.** $|2x + 3| \leq 8$

3. $\left| x - \frac{1}{2} \right| \leq 11$ **4.** $|-4x + 5| > 13$

5. $|4 - 5x| \geq 7$ **6.** $\left|\frac{x + 6}{7}\right| > 1$

7. $\left|\frac{5x - 3}{4}\right| \leq 2$ **8.** $|x - 1| + 2 < 3$

In 9–41, write each expression in simplest radical form. Variables representing indexes are positive integers. Variables in the radicand of an even index are non-negative. Variables occurring in the denominator of a fraction are non-zero.

9. $\sqrt{128}$ **10.** $\sqrt{\frac{3}{4}}$ **11.** $\sqrt{75} + \sqrt{300}$

12. $\sqrt{45} - \sqrt{20}$ **13.** $3\sqrt{12} + 5\sqrt{27}$ **14.** $\frac{1}{2}\sqrt{72} + \frac{1}{3}\sqrt{18}$

15. $\sqrt{24} + \sqrt{20} + \sqrt{54}$ **16.** $\sqrt{8} \times \sqrt{18}$ **17.** $3\sqrt{5} \times \sqrt{125}$

18. $\sqrt{2}(3 + \sqrt{8})$ **19.** $\sqrt{40}(\sqrt{10} - \sqrt{20})$ **20.** $6\sqrt{2}(1 + \sqrt{50})$

21. $(1 + \sqrt{2})(1 - \sqrt{2})$ **22.** $(5 - \sqrt{3})(5 + \sqrt{3})$ **23.** $(2 + \sqrt{3})(1 + \sqrt{3})$

24. $\frac{\sqrt{24}}{\sqrt{12}}$ **25.** $\frac{10 + \sqrt{5}}{\sqrt{5}}$ **26.** $\frac{14}{\sqrt{8} + 1}$

27. $\frac{\sqrt{3}}{4 - \sqrt{3}}$ **28.** $\sqrt{9a} + \sqrt{25a}$ **29.** $\sqrt{8b^3} + \sqrt{50b^3}$

30. $\sqrt[4]{12x^5}$ **31.** $\sqrt{\frac{a}{b}}$ **32.** $16\sqrt{x^5} - \sqrt{16x^5}$

33. $\sqrt[4]{256x^4} - \sqrt[4]{36x^4}$ **34.** $\sqrt{x^3y^4} \cdot \sqrt{xy}$ **35.** $\sqrt[3]{18a^5} \cdot \sqrt[3]{3a^4}$

36. $\frac{\sqrt{64a^5}}{\sqrt{4a}}$ **37.** $\frac{\sqrt[3]{27b^7}}{\sqrt[3]{3b}}$ **38.** $\frac{1}{3 - \sqrt{x}}$

39. $\frac{4 - \sqrt{a}}{4 + \sqrt{a}}$ **40.** $\frac{\sqrt[n]{a^{n+4}}}{\sqrt[n]{a^4}}$

41. $\sqrt[n]{x^{n+2}} \cdot \sqrt[n]{x^{n-2}}$, where n is an integer > 2

In 42–49, solve each equation and check.

42. $\sqrt{2x - 1} = 5$ **43.** $\sqrt{3x - 1} = \sqrt{2x + 4}$

44. $3 + \sqrt{x + 7} = 8$ **45.** $8 + \sqrt{x - 5} = 11$

46. $y = \sqrt{7y - 12}$ **47.** $b = 3 + \sqrt{29 - 5b}$

48. $x + 1 + \sqrt{6 - 2x} = 0$ **49.** $\sqrt{a^2 - 2} = \sqrt{a + 4}$

50. The lengths of the sides of a triangle are 8 feet, $\sqrt{75}$ feet, and $(4 + \sqrt{12})$ feet. Express the perimeter of the triangle is simplest radical form.

51. The length of each leg of an isosceles right triangle is $2\sqrt{2}$ meters.

  a. Find the length of the hypotenuse of the triangle in simplest radical form.

  b. Find the perimeter of the triangle in simplest radical form.

52. Show that x, $\sqrt{2x+1}$, and $x+1$ can be the lengths of the sides of a right triangle for all $x > 0$.

53. The area of a rectangle is 12 square feet and the length is $2 + \sqrt{3}$ feet.

    a. Find the width of the rectangle is simplest radical form.

    b. Find the perimeter of the rectangle in simplest radical form.

Exploration

As mentioned in the chapter opener, Euclid proved that $\sqrt{2}$ is irrational. Following is a proof by contradiction based on Euclid's reasoning.

    Assume that $\sqrt{2}$ is a rational number. Then we can write $\sqrt{2} = \frac{a}{b}$ with a and b integers and $b \neq 0$. Also assume that $\frac{a}{b}$ is a fraction in simplest form. Therefore:

$$2 = \frac{a^2}{b^2} \text{ or } a^2 = 2b^2$$

If $a^2 = 2b^2$, then the square of a is an even number. Since $a^2 = a \cdot a$ and a^2 is even, then a must also be even. An even number can be written as two times some other integer, so we can write $a = 2k$. Substituting into the original equation $2 = \frac{a^2}{b^2}$:

$$2 = \frac{(2k)^2}{b^2}$$
$$2 = \frac{4k^2}{b^2}$$
$$2b^2 = 4k^2$$
$$b^2 = 2k^2$$

This means that b is also even. Therefore, $\frac{a}{b}$ is *not* in simplest form because a and b have a common factor, 2. This contradicts the assumption that $\frac{a}{b}$ is in simplest form, so $\sqrt{2}$ must be irrational.

    There are many other proofs that $\sqrt{2}$ is irrational. Research one of these proofs and discuss your findings with the class.

CUMULATIVE REVIEW   CHAPTERS 1–3

Part I

Answer all questions in this part. Each correct answer will receive 2 credits. No partial credit will be allowed.

    1. Which of the following is not a rational number?

      (1) $\frac{1}{2}$ (2) $0.\overline{25}$ (3) $\sqrt{0.04}$ (4) $\sqrt{0.4}$

2. When $3a^2 - 5a$ is subtracted from $a^2 + 4$, the difference is
(1) $-3a^2 + 5a + 4$ (3) $-2a^2 + 5a + 4$
(2) $-2a^2 - 5a + 4$ (4) $2a^2 - 5a - 4$

3. When factored completely, $12a^2 - 3a$ is equal to
(1) $3a(4a)$ (3) $3(4a^2 - a)$
(2) $a(12a - 3)$ (4) $3a(4a - 1)$

4. The factors of $a^2 + 5a - 6$ are
(1) $(a + 3)(a - 2)$ (3) $(a + 6)(a - 1)$
(2) $(a - 3)(a + 2)$ (4) $(a - 6)(a + 1)$

5. If $4(x + 6) = x - 12$, then x is equal to
(1) -12 (2) -6 (3) 6 (4) 12

6. The solution set of $|y - 3| = 7$ is
(1) $\{10\}$ (2) $\{4, -10\}$ (3) $\{-4, 10\}$ (4) $\{-4\}$

7. The solution set of $3x^2 - x = 2$ is
(1) $\{1, 2\}$ (2) $\left\{\frac{2}{3}, 1\right\}$ (3) $\left\{-\frac{2}{3}, -1\right\}$ (4) $\left\{-\frac{2}{3}, 1\right\}$

8. In simplest form, $\dfrac{1 + \frac{1}{2}}{1 - \frac{1}{4}}$ is equal to

(1) $\frac{3}{8}$ (2) $\frac{1}{2}$ (3) $\frac{2}{3}$ (4) 2

9. If $\frac{3}{x} - 5 > \frac{8}{x}$, then
(1) $-1 < x < 1$ (3) $0 < x < 1$
(2) $x > -1$ or $x < -1$ (4) $-1 < x < 0$

10. In simplest form, $\sqrt{24} + \sqrt{150}$ is equal to
(1) $7\sqrt{6}$ (2) $7\sqrt{12}$ (3) $10\sqrt{6}$ (4) $\sqrt{174}$

Part II

Answer all questions in this part. Each correct answer will receive 2 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

11. Divide and write the quotient in simplest form.

$$\frac{a^2 - 16}{a^2 - a - 12} \div \frac{a^2 + 4a}{2a}$$

12. Solve and check: $2x^2 - 9x = 5$.

Part III

Answer all questions in this part. Each correct answer will receive 4 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

13. Find the solution set of the inequality $|3b + 6| < 7$ and graph the solution on the number line.

14. On a test, the number of questions that Tyler answered correctly was 4 more than twice the number that he answered incorrectly. If the ratio of correct answers to incorrect answers is $8 : 3$, how many questions did Tyler answer correctly?

Part IV

Answer all questions in this part. Each correct answer will receive 6 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

15. On her way to work, Rachel travels two miles on local roads and 12 miles on the highway. Her speed on the highway is twice her speed on local roads. Her driving time for the trip to work is 16 minutes. What is Rachel's rate of speed for each part of the trip?

16. The length of Bill's garden is 2 yards more than four times the width. The area of the garden is 30 square yards. What are the dimensions of the garden?